

From de Broglie Waves to Option Pricing: A Wave Mechanics Approach to Financial Derivatives

— Working Paper —

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A quantum inspired financial modelling

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Abstract

The Black–Scholes model, cornerstone of modern option pricing, is mathematically equivalent to a heat diffusion equation. It is well known that the heat equation is related to the Schrödinger equation of quantum mechanics through a Wick rotation — the substitution of real time by imaginary time. The existing literature on quantum finance has exploited this connection, but systematically eliminates the imaginary unit i in the process, thereby discarding the oscillatory and wave-like features that make quantum mechanics so distinctive.

In this paper, we take a different path. Inspired by Louis de Broglie’s original hypothesis that every particle carries an associated wave, we construct a *genuine* wave mechanics for financial assets, in which the imaginary unit i plays a structural role rather than being rotated away. We associate to each underlying asset a financial wave function $\psi(x, t)$ governed by a Schrödinger-type equation, and we show that wave phenomena (interference, tunnelling, diffraction and wave-function collapse) provide natural explanations for empirically observed market features such as the volatility smile, barrier option mispricing, false breakouts and post-announcement dynamics. We derive a closed-form pricing formula for European options that decomposes into classical Black–Scholes components plus an interference correction term, and we compute the full set of associated hedging sensitivities, including three novel “Greeks” that have no classical counterpart: the Planck Greek Ξ , the phase Greek Φ and the coherence Greek κ . We verify that the model reduces exactly to Black–Scholes in the appropriate classical limit.

Keywords: Option pricing, quantum finance, de Broglie waves, Schrödinger equation, volatility smile, wave interference, hedging Greeks, financial wave mechanics.

JEL Classification: G13, C65, G12.

1 Introduction

The history of mathematical finance is, in many ways, a history of borrowed physics. The foundational work of Louis Bachelier in 1900 on the theory of speculation predates Einstein's treatment of Brownian motion by five years. The Black–Scholes–Merton framework, which earned Scholes and Merton the 1997 Nobel Memorial Prize in Economics, rests upon the same stochastic calculus that physicists use to describe diffusion phenomena. The deep structural parallel between the two fields is not a coincidence: both deal with systems subject to random fluctuations, both seek to extract deterministic pricing or evolution laws from underlying stochasticity, and both employ partial differential equations as their central analytical tool.

Yet the borrowing has been selective. Finance has drawn heavily from classical statistical mechanics and the theory of diffusion, but it has largely ignored the other great revolution of twentieth-century physics: quantum mechanics. This is surprising since the mathematical connection between the two frameworks is remarkably tight. The Black–Scholes equation is formally a heat equation, and the heat equation is related to the Schrödinger equation — the master equation of quantum mechanics — through a simple but profound substitution known as a *Wick rotation*, in which real time t is replaced by imaginary time it . In other words, the Black–Scholes equation is the Schrödinger equation with the imaginary unit i removed.

This observation, first explored systematically by Baaquie (1997) and Haven (2002), has given rise to a growing literature on *quantum finance*. Researchers have recast Black–Scholes in the language of Hilbert spaces and Hamiltonians (Contreras et al., 2010; Baaquie, 2004), solved multi-asset options using quantum-mechanical techniques (Baaquie, 2023), applied Feynman path integrals to exotic and path-dependent derivatives (Baaquie, 2002), and even proposed quantum algorithms for option pricing on quantum computers (QNUTE, 2025). The Ivancevic option pricing model uses a non-linear Schrödinger equation to describe option price dynamics (Ivancevic, 2024), while Orrell (2020) has proposed a quantum walk model that can run on photonic hardware.

However, virtually all of these approaches share a common feature: they work in *imaginary time*. The Wick rotation that connects Black–Scholes to Schrödinger is performed, the mathematical machinery of quantum mechanics is deployed, and useful results are obtained — but the imaginary unit i , which is the very essence of quantum wave mechanics, is systematically eliminated. As Arioli & Valente (2023) have pointed out, the Black–Scholes equation uses no imaginary numbers, and since quantum phenomena like superposition and interference arise precisely from the complex phase of the wave function, the numerical successes of quantum finance must stem from effects other than genuinely quantum ones.

In this paper, we propose to take the opposite approach. Rather than starting from Black–Scholes and recovering a Schrödinger equation through mathematical manipulation, we start from first principles — specifically, from Louis de Broglie's 1924 hypothesis — and construct a *genuine* financial wave mechanics in which i is structural, not cosmetic. Our programme mirrors the historical development of quantum mechanics itself. De Broglie proposed that every material particle carries an associated wave, characterised by a wavelength inversely proportional to its momentum. Schrödinger then asked: what equation governs the evolution of this wave? The answer was the celebrated Schrödinger equation, in which i appears necessarily, not as an artefact.

We follow the same path. We associate to each financial asset a *financial wavelength* through a de Broglie-type relation, construct a Schrödinger equation for the resulting financial wave function, and explore the consequences. Because i is preserved, the propagator of our equation is oscillatory rather than purely diffusive, and a rich family of wave phenomena becomes available: interference between competing market signals, tunnelling through price barriers, diffraction upon breakout from a trading range and wave-function collapse upon the arrival of definitive information. These phenomena provide natural and intuitive explanations for market features that the classical framework either cannot capture or must accommodate through ad hoc adjustments.

The paper is organised as follows. Section 2 develops the de Broglie analogy and establishes the dictionary between quantum and financial quantities. Section 3 derives the financial Schrödinger

equation and discusses its propagator. Section 4 explores the four principal wave phenomena and their financial interpretations. Section 5 derives the quantum pricing formula for European options. Section 6 computes the full set of hedging Greeks, including the three novel ones. Section 7 verifies the classical limit. Section 8 discusses implications, limitations and directions for future research.

2 The Financial de Broglie Hypothesis

2.1 Historical context

In his 1924 doctoral thesis, Louis de Broglie made one of the most audacious proposals in the history of physics. Building on Einstein’s demonstration that light, traditionally understood as a wave, also behaves as a particle (the photon), de Broglie proposed the converse: matter, traditionally understood as composed of particles, also carries an associated wave. The wavelength of this *matter wave* is given by the now-famous relation

$$\lambda = \frac{h}{p} \quad (1)$$

where h is Planck’s constant and $p = mv$ is the momentum of the particle. The proposal was initially met with scepticism. Einstein is said to have called it “a first feeble ray of light on this worst of our physics enigmas”. But it was confirmed experimentally by Davisson and Germer in 1927 and became one of the pillars of quantum mechanics.

The key insight of de Broglie was that the wavelength is *inversely* proportional to the momentum. A heavy and fast-moving object has an extremely short wavelength, so its wave nature is imperceptible and it behaves classically. A light and slow-moving object has a long wavelength, and its wave nature dominates. The transition between these two regimes, from classical to quantum behaviour, is governed by the ratio of the wavelength to the characteristic length scale of the problem.

We now propose to transplant this idea, with appropriate modifications, into the world of finance.

2.2 The physics–finance dictionary

The starting point is to identify the correct correspondences between physical and financial quantities. These correspondences must be more than formal analogies. They must carry the right physical intuitions. We work in the space of *log-prices*, defining $x = \ln(S/S_0)$ where S is the price of the underlying asset and S_0 is a reference price (typically the current spot price). The log-price is the natural coordinate in finance, just as position is the natural coordinate in mechanics. Returns are additive in log-prices.

The first correspondence is between mass and volatility. In physics, the mass of a particle measures its inertia, its resistance to changes in velocity. A heavy particle is hard to move, a light particle responds readily to forces. In finance, the analogous quantity is the *inverse of the variance of returns*. An asset with low volatility is “heavy”, difficult to move from its current price level, while a highly volatile asset is “light”, easily pushed around by market forces. We therefore define the *financial mass* as

$$m_f = \frac{1}{\sigma^2} \quad (2)$$

where σ is the volatility of the underlying asset. This identification is not new. It appears implicitly in the quantum finance literature, but it is important to recognise its intuitive content. A blue-chip stock with $\sigma \approx 15\%$ has a financial mass of roughly 44, while a volatile tech stock with $\sigma \approx 50\%$ has a mass of only 4. The blue-chip is eleven times “heavier” than the tech stock in this precise sense.

The second correspondence is between momentum and drift-weighted-by-mass. In classical mechanics, momentum is $p = mv$, where v is velocity. In our financial setting, the velocity is the drift

of the log-price μ (the expected rate of return minus half the variance, or simply the instantaneous drift), and the financial momentum is

$$p_f = m_f \cdot \mu = \frac{\mu}{\sigma^2} \quad (3)$$

This quantity is dimensionally the ratio of drift to variance, which is recognisable to any quantitative analyst as being closely related to the *Sharpe ratio* (up to a factor of σ). A market with strong directional conviction and low volatility has large financial momentum. A directionless and highly volatile market has near-zero momentum.

The third correspondence concerns Planck's constant. In physics, \hbar sets the scale of quantum effects. When the *action* of a system (a quantity with dimensions of energy \times time) is of the order of \hbar , quantum effects dominate. When it is much larger, classical mechanics suffices. We introduce a *financial Planck constant* \hbar_f that plays the same role. It sets the scale at which wave-like, non-classical effects become important in the market. Following Haven (2002), who linked this constant to the presence of arbitrage opportunities, we interpret \hbar_f as a measure of *market imperfection*. When $\hbar_f \rightarrow 0$, the market is perfectly efficient, no wave effects are present and the classical (Black–Scholes) description is exact. When \hbar_f is appreciably different from zero, market imperfections (transaction costs, information asymmetry, microstructure noise, behavioural biases) generate detectable wave phenomena.

2.3 The financial de Broglie relation

With these correspondences in hand, we can state the financial de Broglie relation by direct analogy with equation (1):

$$\lambda_f = \frac{\hbar_f}{p_f} = \frac{\hbar_f \sigma^2}{\mu} \quad (4)$$

This is the *financial wavelength* of the underlying asset. Its interpretation is rich and illuminating for practitioners. Consider three limiting regimes.

When the drift is large and the volatility is small — the market is moving decisively in one direction with little noise — the financial wavelength is very short. In this regime, the asset behaves like a classical particle. It moves along a well defined trajectory, the probability distribution of future prices is tightly concentrated around the drift, and the Black–Scholes model provides an excellent description. There is nothing to be gained from a wave-mechanical approach.

When the volatility is large and the drift is small — the market is directionless and dominated by conflicting signals — the financial wavelength becomes large. In this regime, the wave nature of the asset manifests itself. The probability distribution of future prices develops oscillatory features (interference fringes), barriers become semi-transparent (tunnelling) and narrow trading ranges produce diffraction patterns upon breakout. This is precisely the regime where Black–Scholes is known to perform poorly and where empirical corrections (the volatility smile, jump-diffusion models, stochastic volatility) are needed.

When the drift approaches zero — the asset is a pure random walk with no directional tendency — the financial wavelength diverges. The asset is completely “delocalised” in price space, analogous to a free particle with zero momentum in quantum mechanics. This extreme case represents maximum uncertainty and maximum quantum character.

The beauty of the de Broglie relation is that it provides a single, dimensionless criterion for assessing when wave effects matter. If we define the relevant length scale L as the width of the price region of interest (i.e. the distance between the current price and the strike, or the width of a trading range), then wave effects are important when $\lambda_f \gtrsim L$ and negligible when $\lambda_f \ll L$. This gives practitioners a concrete, computable diagnostic.

2.4 The financial plane wave

Following de Broglie, we associate to the underlying asset a plane wave:

$$\psi(x, t) = A \exp\left[i(k_f x - \omega_f t)\right] \quad (5)$$

where $k_f = p_f/\hbar_f$ is the *financial wave number* and $\omega_f = E_f/\hbar_f$ is the *financial angular frequency*, with E_f the financial energy, to be specified in the next section. The imaginary unit i appears here in a structural and ineliminable way. It encodes the *phase* of the wave, i.e. the timing and rhythm of the oscillations. It is the phase that produces the interference effects that are the hallmark of wave mechanics.

Two assets can have the same probability amplitude $|A|$ at a given point in price space but different phases. In a purely diffusive framework such as Black–Scholes, the phase is invisible and the two assets look identical. In our wave framework, the phase difference between the two determines whether their combined probability density is enhanced (constructive interference) or suppressed (destructive interference) at each point. This is the fundamental new ingredient.

3 The Financial Schrödinger Equation

3.1 Derivation from the de Broglie wave

Schrödinger’s derivation of his equation in 1926 can be understood as follows. He started with de Broglie’s plane wave (5) and the classical energy–momentum relation for a particle of mass m in a potential $V(x)$:

$$E = \frac{p^2}{2m} + V(x) = \frac{1}{2}m v^2 + V(x) \quad (6)$$

He then observed that the operators $i\hbar\partial/\partial t$ and $-i\hbar\partial/\partial x$, when applied to the plane wave, yield $E\psi$ and $p\psi$ respectively. Replacing E and p by these operators in the energy–momentum relation and applying the result to ψ , he obtained the time-dependent Schrödinger equation.

We follow exactly the same procedure. The financial energy–momentum relation is

$$E_f = \frac{p_f^2}{2m_f} + V(x, t) = \frac{\mu^2}{2\sigma^2} + V(x, t) \quad (7)$$

where $V(x, t)$ is a *financial potential* representing the external forces acting on the price (barriers, market microstructure, macroeconomic pressure and so forth). The kinetic term $\mu^2/(2\sigma^2)$ represents the “kinetic energy” of the price movement. In some sense, the cost of sustaining a given drift against a given level of randomness.

Applying the operator substitution and using $m_f = 1/\sigma^2$, we obtain the **financial Schrödinger equation**:

$$\boxed{i\hbar_f \frac{\partial\psi}{\partial t} = -\frac{\hbar_f^2 \sigma^2}{2} \frac{\partial^2\psi}{\partial x^2} + V(x, t)\psi} \quad (8)$$

Each term carries a clear financial interpretation. The left hand side, $i\hbar_f \partial\psi/\partial t$, describes the temporal evolution of the financial wave function, and the factor of i ensures that this evolution is oscillatory (wave-like) rather than dissipative (diffusion-like). The first term on the right, $-(\hbar_f^2 \sigma^2/2) \partial^2\psi/\partial x^2$, is the kinetic term. It governs how the probability amplitude spreads across price space, with the rate of spreading determined by the volatility σ . The second term, $V(x, t)\psi$, is the potential term. It describes how external forces, i.e. the “landscape” of the market, deflect, confine or accelerate the wave.

3.2 The connection to Black–Scholes

The relationship between our equation and the classical Black–Scholes framework is elegant and instructive. If we perform a Wick rotation, i.e. replacing t by it or equivalently setting $\hbar_f = -i$, the factor of i on the left-hand side of equation (8) becomes $i \times (-i) = 1$, and the equation reduces to

$$\frac{\partial \psi}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} - V(x, t) \psi \quad (9)$$

which is the heat equation with a source term, precisely the form of the Black–Scholes equation after the standard change of variables. The Black–Scholes world is therefore the “imaginary-time” or “diffusive” limit of our wave-mechanical world. Everything that Black–Scholes describes is contained in our model as a special case. But our model also contains oscillatory phenomena that Black–Scholes cannot see, because those phenomena are carried by the phase of ψ , which the Wick rotation destroys.

The analogy can be pushed further. In quantum mechanics, the Wick-rotated Schrödinger equation is used to study thermodynamic equilibrium (the partition function in statistical mechanics). In finance, the Wick-rotated equation gives the equilibrium, arbitrage-free price. Our unrotated equation, by contrast, describes a market that is *not necessarily in equilibrium*, i.e. a market in which arbitrage opportunities, information asymmetries and microstructure effects create wave-like disturbances in the price distribution. This is the regime that Haven (2002) identified as requiring $\hbar_f \neq 0$.

3.3 The quantum financial propagator

The free-particle propagator of equation (8) with $V = 0$ is obtained by standard methods:

$$K(x, T; x', 0) = \sqrt{\frac{m_f}{2\pi i \hbar_f T}} \exp\left(\frac{i m_f (x - x')^2}{2 \hbar_f T}\right) \quad (10)$$

The crucial difference with the Black–Scholes heat propagator is the factor of i in both the prefactor and the exponent. The heat propagator is a real Gaussian that spreads and decays. It describes a probability distribution that becomes progressively wider and flatter over time. Our propagator is a *complex* Gaussian that oscillates as well as spreads. It describes a probability amplitude whose modulus spreads but whose phase encodes information about the “velocity” (drift) of the distribution.

Given an initial wave function $\psi_0(x')$, the wave function at any later time is given by convolution with the propagator:

$$\psi(x, T) = \int_{-\infty}^{+\infty} K(x, T; x', 0) \psi_0(x') dx' \quad (11)$$

For a Gaussian initial state centred at x_0 with width σ_0 and initial momentum p_0 ,

$$\psi_0(x) = \left(\frac{1}{2\pi\sigma_0^2}\right)^{1/4} \exp\left(-\frac{(x-x_0)^2}{4\sigma_0^2} + \frac{i p_0 x}{\hbar_f}\right) \quad (12)$$

the convolution can be evaluated exactly. The resulting wave function at time T is again Gaussian, but with a width that has grown to

$$\sigma_{\text{eff}}^2(T) = \sigma_0^2 + \frac{\hbar_f^2 T^2 \sigma^2}{4\sigma_0^2} \quad (13)$$

a centre that has shifted to $x_0 + (p_0/m_f)T = x_0 + \mu T$ (the packet follows the classical drift) and a complex phase that accumulates over time. The broadening formula (13) is the quantum analogue of the classical result $\sigma^2 T$ for the variance of a diffusion process. But it contains an additional \hbar_f -dependent term that encodes the *dispersive* spreading of the wave packet, i.e. the fact that different momentum components of the wave travel at different speeds.

4 Wave Phenomena in Financial Markets

We now turn to the physical heart of the paper. The wave phenomena that become accessible when i is preserved and the financial Schrödinger equation is taken seriously as a wave equation. We discuss four phenomena — interference, tunnelling, diffraction and wave-function collapse — each of which has a clear and illuminating financial interpretation.

4.1 Interference: when market signals collide

Interference is perhaps the most iconic quantum phenomenon. When two waves meet, they do not simply “average out”. Instead, their amplitudes add and the resulting intensity (the square of the total amplitude) contains an oscillatory *cross term* that can be either positive or negative depending on the relative phase of the two waves. Where the waves are in phase, they reinforce each other (constructive interference), and where they are out of phase, they cancel (destructive interference).

The financial analogue is natural. Two sources of information, such as a bullish macroeconomic signal and a bearish sector-specific signal, generate two “market waves” ψ_1 and ψ_2 , each carrying its own momentum (drift expectation) and phase (timing). The total state of the market is the superposition $\psi = w_1\psi_1 + w_2\psi_2$, where w_1 and w_2 are complex weights reflecting the relative strength and credibility of each signal, with $|w_1|^2 + |w_2|^2 = 1$.

The probability density of the log-price at maturity is then

$$\rho_Q(x, T) = |\psi|^2 = |w_1|^2|\psi_1|^2 + |w_2|^2|\psi_2|^2 + 2\operatorname{Re}(w_1^* w_2 \psi_1^* \psi_2) \quad (14)$$

The first two terms are the individual probability densities, as we would obtain in a classical mixture. The third term is the *interference term*. For Gaussian wave packets with wave numbers k_1 and k_2 respectively, this term oscillates as

$$\text{Interference} \propto \cos(\Delta k \cdot x - \Delta \omega \cdot T + \varphi_0) \quad (15)$$

where $\Delta k = k_1 - k_2$ is the difference in wave numbers, $\Delta \omega = \omega_1 - \omega_2$ is the difference in frequencies and φ_0 is an initial phase offset. The fringe spacing in log-price space is $2\pi/|\Delta k|$.

The financial consequences are striking. In regions where the interference is constructive, the probability density is *higher* than the classical mixture would predict. The price is anomalously likely to be found in these regions. These are the natural candidates for what practitioners call *support and resistance levels*, price levels at which the market seems to “stick”, not because of any single identifiable force, but because of the constructive superposition of multiple market signals. Conversely, in regions of destructive interference, the probability density is anomalously low. The price tends to jump through these levels rapidly, producing *gaps* and zones of low liquidity.

Importantly, the fringe spacing depends on Δk , which is proportional to the difference in momentum between the two signals. When the market receives two signals that are sharply contradictory (Δk large), the fringes are closely spaced and the microstructure of the price distribution is complex and rapidly oscillating. When the signals are nearly aligned (Δk small), the fringes are wide and the distribution is smooth. The classical limit is approached. This provides a natural, parameter-free mechanism for the transition between “turbulent” and “calm” market regimes.

4.2 Tunnelling: breaching the barrier

In classical mechanics, a particle encountering a potential barrier higher than its kinetic energy is reflected with certainty. In quantum mechanics, the particle has a non-zero probability of *tunnelling through* the barrier, with the transmission probability decaying exponentially with the barrier’s height and width. This phenomenon has no classical analogue and is responsible for, among other things, nuclear fusion in stars and the operation of tunnel diodes.

In our financial framework, a potential barrier $V(x)$ represents a strong support or resistance level, i.e. a price region where significant opposing forces exist (accumulated limit orders, psychological thresholds, technical levels). Consider a rectangular barrier of height V_0 and width L in log-price space, centred at the barrier price $x_B = \ln(B/S_0)$. The financial Schrödinger equation (8) has a standard solution in three regions: an incoming and reflected wave before the barrier, an evanescent (exponentially decaying) wave inside the barrier and a transmitted wave beyond it.

The transmission probability is

$$|T|^2 \approx \exp\left(-\frac{2L}{\hbar_f \sigma} \sqrt{2(V_0 - E_f)}\right) \quad (16)$$

where E_f is the kinetic energy of the incoming wave. Several features of this result are financially significant. The tunnelling probability is non-zero even when the price “should not” be able to breach the barrier based on its current drift and volatility. The wave can leak through. This probability increases when \hbar_f is large (a highly imperfect market where information frictions and microstructure effects are strong), when σ is large (a volatile asset), and when the barrier is thin (L small¹). Conversely, a thick barrier (L large²) is exponentially more robust.

For option pricing, the tunnelling effect is most relevant for *barrier options*. In the classical framework, a knock-out option dies the instant the price touches the barrier. In our wave framework, the wave function penetrates partially into the barrier, so there is a non-zero probability that the price “touches” the barrier and returns without the option truly dying. This gives a quantum barrier option price that is *higher* than the classical price, which is consistent with the well documented empirical observation that barrier options are systematically underpriced by classical models near the barrier.

4.3 Diffraction: the physics of breakouts

When a wave passes through a narrow opening, it does not continue in a straight line but spreads out. It *diffracts*. The angular spread of the diffracted wave is proportional to λ/d , where λ is the wavelength and d is the width of the opening. When $d \gg \lambda$, diffraction is negligible and the wave propagates almost rectilinearly. When $d \sim \lambda$, diffraction is strong and the wave spreads in all directions.

The financial analogue is a *breakout from a narrow trading range*. When the price is confined in a channel of width d in log-price space (a squeeze, a consolidation, a narrow Bollinger band) and then “breaks out”, the subsequent behaviour of the price depends on the ratio λ_f/d . If $\lambda_f \ll d$ (large momentum, low volatility relative to the range width), the breakout is clean and directional, i.e. the price moves decisively in the breakout direction and the classical picture is adequate. But if $\lambda_f \sim d$ (small momentum, high volatility relative to the range width), the diffraction angle $\theta \sim \lambda_f/d$ is large. The price spreads out in multiple directions after the breakout, producing what practitioners call a *false breakout*.

False breakouts are the bane of technical traders. In the classical framework, they are attributed to “noise” or “randomness” without further structure. In our wave framework, they are a manifestation of diffraction and they have a *specific structure*. The diffracted wave produces an intensity pattern with a central peak in the breakout direction and subsidiary peaks at predictable angles, separated by dark fringes (price levels that are unlikely to be visited). In principle, this structure is testable.

4.4 Wave-function collapse: the impact of news

In quantum mechanics, the wave function ψ evolves smoothly according to the Schrödinger equation until a *measurement* is performed, at which point the wave function “collapses” to an eigenstate

¹ L small — a support or resistance level confirmed by few data points

² L large — a level confirmed by many technical indicators and heavy order flow

of the measured observable. Before the measurement, the system is in a superposition of possible states. After the measurement, it is in a definite state.

The financial analogue is the arrival of *definitive information*, such as an earnings announcement, a central bank decision, an election result. Before the announcement, the market is in a superposition of possible scenarios, each with an associated probability amplitude:

$$\psi_{\text{before}} = \sum_n c_n \phi_n \quad (17)$$

where ϕ_n represents the price distribution conditional on scenario n and $|c_n|^2$ is the probability of that scenario. After the announcement, one scenario k is realised and the wave function collapses:

$$\psi_{\text{after}} = \phi_k \quad (18)$$

The probability that scenario k is realised is $|c_k|^2$ and the post-collapse state ϕ_k is typically a narrow, localised wave packet centred on the price implied by the realised scenario. From this localised state, a new wave then begins to spread (the post-announcement dynamics) and the cycle begins again.

This framework captures something that Black–Scholes cannot. The *discontinuous* nature of information arrival. The collapse is instantaneous and non-unitary (it cannot be described by the Schrödinger equation alone), just as in quantum mechanics. It provides a natural model for the sudden volatility compression observed after announcements (the wave function goes from broad to narrow) and for the subsequent volatility expansion (the narrow wave packet begins to spread again).

5 Quantum Option Pricing

5.1 The two-signal model

We now specialise to a concrete model that captures the essential physics while remaining analytically tractable. We consider a European call option with strike K and maturity T , written on an underlying asset whose current log-price is $x_0 = 0$ (i.e. S_0 is the reference price). The market contains two competing signals, modelled as Gaussian wave packets ψ_1 and ψ_2 with respective wave numbers k_1 and k_2 , initial widths σ_0 , and weights w_1, w_2 with $w_1^2 + w_2^2 = 1$ (we take the weights to be real for simplicity). The total wave function is $\psi = w_1\psi_1 + w_2\psi_2$.

Each component evolves freely according to equation (8) with $V = 0$. We discuss potential effects separately. At maturity T , the risk-neutral probability density is $\rho_Q(x, T) = |\psi(x, T)|^2$ and the call price is

$$C_Q = e^{-rT} \int_{-\infty}^{+\infty} \rho_Q(x, T) \max(S_0 e^x - K, 0) dx \quad (19)$$

5.2 Decomposition of the price

Expanding ρ_Q as in equation (14), the call price decomposes into three terms:

$$\boxed{C_Q = w_1^2 C_1 + w_2^2 C_2 + C_{\text{int}}} \quad (20)$$

where C_1 and C_2 are Black–Scholes-type prices corresponding to each signal, and C_{int} is the *interference correction*.

The individual prices C_j ($j = 1, 2$) take the familiar Black–Scholes form, evaluated with the effective parameters of each wave packet:

$$C_j = S_0 e^{(\mu_j - r)T} \mathcal{N}(d_{1,j}) - K e^{-rT} \mathcal{N}(d_{2,j}) \quad (21)$$

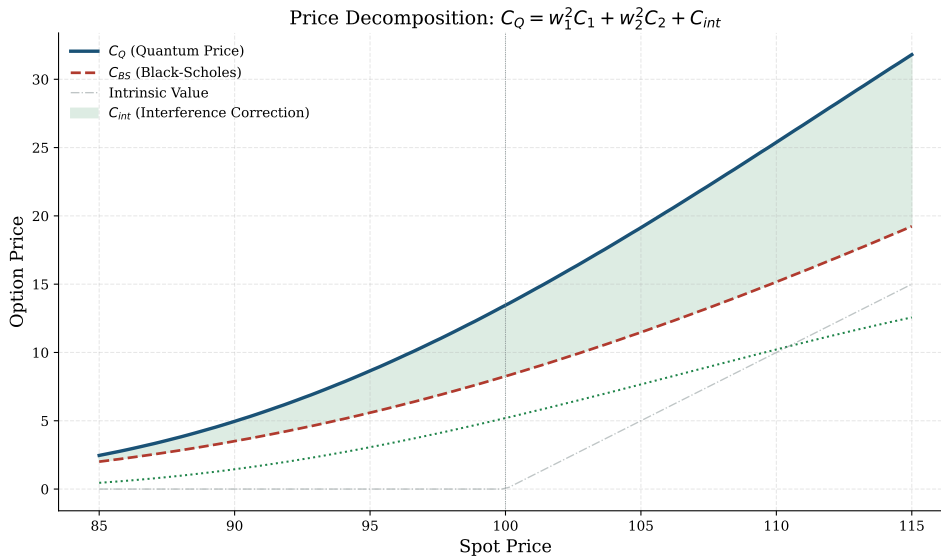


Figure 1: Quantum price decomposition across spot prices. The total quantum price $C_Q = w_1^2 C_1 + w_2^2 C_2 + C_{int}$ is shown in dark blue, the Black-Scholes price in dashed red, and the interference correction as the shaded green region between the two curves. The intrinsic value $(S - K)^+$ is shown for reference.

where $\mu_j = \hbar_f k_j \sigma^2$ is the drift implied by signal j , $\mathcal{N}(\cdot)$ is the standard normal cumulative distribution function, and the d -variables are

$$d_{1,j} = \frac{\ln(S_0/K) + (\mu_j + \frac{1}{2} \sigma_{\text{eff}}^2/T) T}{\sigma_{\text{eff}} \sqrt{T}}, \quad d_{2,j} = d_{1,j} - \frac{\sigma_{\text{eff}} \sqrt{T}}{1} \quad (22)$$

with σ_{eff} given by equation (13).

5.3 The interference term

The genuinely new contribution is the interference term:

$$C_{\text{int}} = 2 w_1 w_2 e^{-rT} \int_{x_K}^{+\infty} |\psi_1(x, T)| |\psi_2(x, T)| \cos(\Delta k x - \Delta \omega T + \varphi_0) (S_0 e^x - K) dx \quad (23)$$

where $x_K = \ln(K/S_0)$ is the log-moneyness of the strike.

For Gaussian wave packets, this integral can be evaluated in closed form. After carrying out the Gaussian integration³, the result is

$$C_{\text{int}} = 2 w_1 w_2 e^{-rT} e^{-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2} \left[S_0 e^{\bar{\mu} T} \text{Re} \left(e^{i \Delta \varphi_K} \mathcal{N}(d_1^{\text{int}}) \right) - K \text{Re} \left(e^{i \Delta \varphi_K} \mathcal{N}(d_2^{\text{int}}) \right) \right] \quad (24)$$

where $\bar{\mu} = (\mu_1 + \mu_2)/2$, $\Delta \varphi_K = \Delta k \cdot x_K - \Delta \omega \cdot T + \varphi_0$ is the interference phase evaluated at the strike, and d_1^{int} , d_2^{int} are modified d -variables.

Several features of this result deserve comment. The factor $e^{-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2}$ is a *coherence factor*. It measures the degree to which the two signals maintain a stable phase relationship. When Δk is large (the signals carry very different momentum, i.e. they disagree strongly about the direction of the market), this factor is exponentially small and the interference washes out. This is the phenomenon of *decoherence*, i.e. strongly contradictory signals destroy each other's coherence, and

³The details are provided in the Appendix 9.4

the result is a classical mixture. Conversely, when Δk is small (the signals are nearly aligned), the coherence factor is close to unity and the interference is maximal.

The factor $\cos(\Delta\varphi_K)$ (embedded in the $\text{Re}(e^{i\Delta\varphi_K} \dots)$ terms) makes the interference correction *oscillatory in the strike*. As we move across the strike space, C_{int} oscillates between positive and negative values. This is the mechanism by which the model generates a *volatility smile*: the implied volatility extracted from C_Q using the Black–Scholes inversion formula will exhibit oscillatory deviations from the flat volatility assumed by Black–Scholes, with the amplitude and frequency of the oscillations controlled by Δk and σ_{eff} respectively.

5.4 The volatility smile as an interference pattern

To make this point more concrete, consider the implied volatility $\sigma_{\text{imp}}(K)$ defined implicitly by

$$C_Q(K) = C_{\text{BS}}(K; \sigma_{\text{imp}}(K)) \quad (25)$$

For small interference corrections, a first-order expansion gives

$$\sigma_{\text{imp}}(K) \approx \sigma_{\text{BS}} + \frac{C_{\text{int}}(K)}{\nu_{\text{BS}}(K)} \quad (26)$$

where $\nu_{\text{BS}}(K) = \partial C_{\text{BS}}/\partial\sigma$ is the Black–Scholes vega. Since $C_{\text{int}}(K)$ oscillates with K (through its dependence on $x_K = \ln(K/S_0)$), the implied volatility inherits this oscillation, producing a smile-like structure. The minimum of the smile occurs near the strike where the interference is maximally destructive, and the wings are lifted where the interference is constructive.

This mechanism is qualitatively different from the standard explanations of the smile (fat tails, stochastic volatility, jumps). It does not require any modification of the underlying process. It arises purely from the *superposition of two signals* within the standard geometric Brownian motion framework, enriched by wave mechanics. In this view, the smile is an interference pattern.

6 The Quantum Greeks

The hedging sensitivities of an option — the “Greeks” — are as important to practitioners as the price itself, since they determine how the option must be dynamically hedged. In this section, we derive the full set of Greeks for our quantum pricing formula and identify three novel Greeks that have no Black–Scholes counterpart.

6.1 Classical Greeks with interference corrections

Each classical Greek receives an additive correction from the interference term. For instance, the **quantum delta** is

$$\Delta_Q = \frac{\partial C_Q}{\partial S_0} = w_1^2 \Delta_1 + w_2^2 \Delta_2 + \Delta_{\text{int}} \quad (27)$$

where Δ_1 and Δ_2 are the Black–Scholes deltas of the individual components, and $\Delta_{\text{int}} = \partial C_{\text{int}}/\partial S_0$.

The interference delta has a remarkable property. It is *not monotonic* in S_0 . The classical delta is a smooth S-shaped curve running from 0 (deep out-of-the-money) to 1 (deep in-the-money). The quantum delta oscillates around this curve, with the amplitude and frequency of the oscillations determined by the interference parameters. A trader using the classical delta to hedge will be systematically over-hedged in some price regions and under-hedged in others, with the pattern being *predictable* (not random) given the interference parameters.

The **quantum gamma** is

$$\Gamma_Q = \frac{\partial^2 C_Q}{\partial S_0^2} = w_1^2 \Gamma_1 + w_2^2 \Gamma_2 + \Gamma_{\text{int}} \quad (28)$$

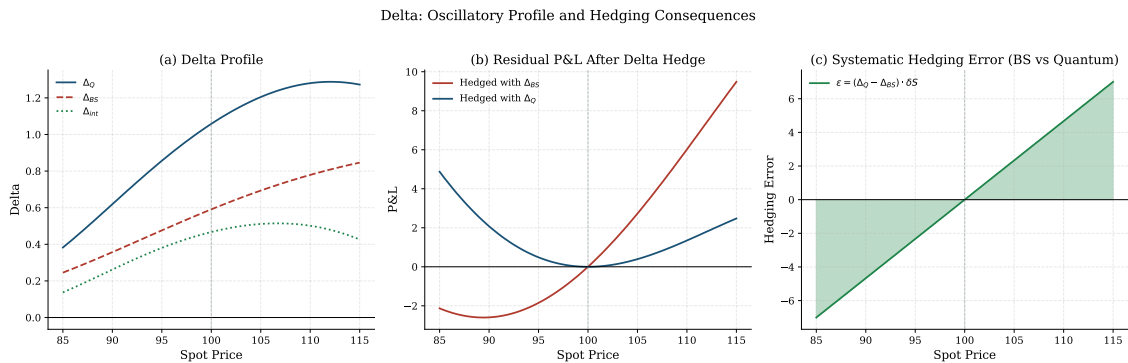


Figure 2: Delta analysis. (a) The quantum delta Δ_Q oscillates around the classical Δ_{BS} , with the interference component Δ_{int} shown as a dotted line. (b) Residual P&L after delta hedging: hedging with Δ_Q leaves a much smaller residual than hedging with Δ_{BS} . (c) The systematic hedging error is oscillatory and predictable.

The interference gamma scales as $(\Delta k)^2$, meaning that it is strongest when the two market signals are most contradictory. This has a clear practical implication. The gamma risk of an option is *underestimated* by the classical model precisely when the market is receiving conflicting information, which is precisely when gamma risk matters most.

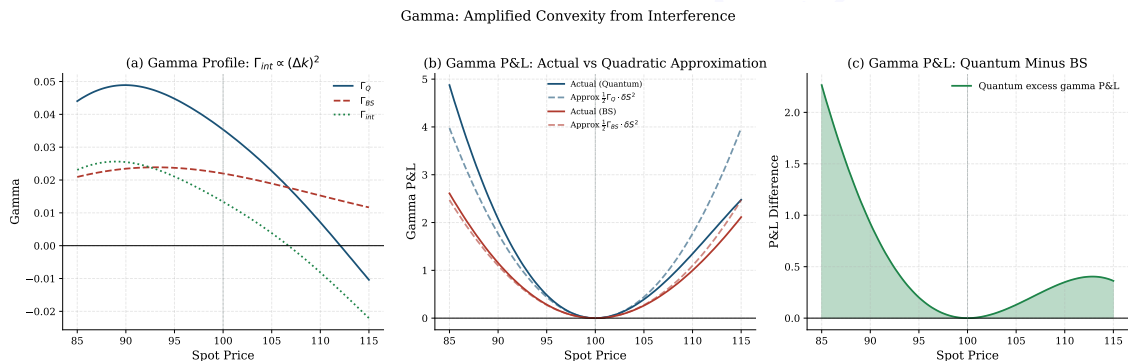


Figure 3: Gamma analysis. (a) The quantum gamma Γ_Q deviates from Γ_{BS} , with $\Gamma_{int} \propto (\Delta k)^2$ amplifying convexity when signals conflict. (b) Realised gamma P&L versus the quadratic approximation $\frac{1}{2}\Gamma \delta S^2$ for both models. (c) The excess gamma P&L of the quantum model over Black–Scholes, showing the additional convexity risk carried by the interference term.

The **quantum vega** is

$$\nu_Q = \frac{\partial C_Q}{\partial \sigma} = w_1^2 \nu_1 + w_2^2 \nu_2 + \nu_{int} \quad (29)$$

The interference vega contains a contribution from the coherence factor $e^{-(\Delta k)^2 \sigma_{eff}^2 / 2}$, whose derivative with respect to σ can be *negative*. This means that, in some regimes, increasing volatility can *reduce* the price of the option by destroying the coherence between the two signals. This is a genuinely surprising prediction. In the Black–Scholes world, vega is always positive for a vanilla option. In our model, the interference vega can partially or (in extreme cases) fully offset the classical vega, leading to a total ν_Q that is reduced or even negative. This occurs when the increase in volatility causes decoherence faster than it increases the classical option value.

The **quantum theta** is

$$\Theta_Q = \frac{\partial C_Q}{\partial T} = w_1^2 \Theta_1 + w_2^2 \Theta_2 + \Theta_{int} \quad (30)$$

Vega: The Decoherence Effect on Volatility Sensitivity

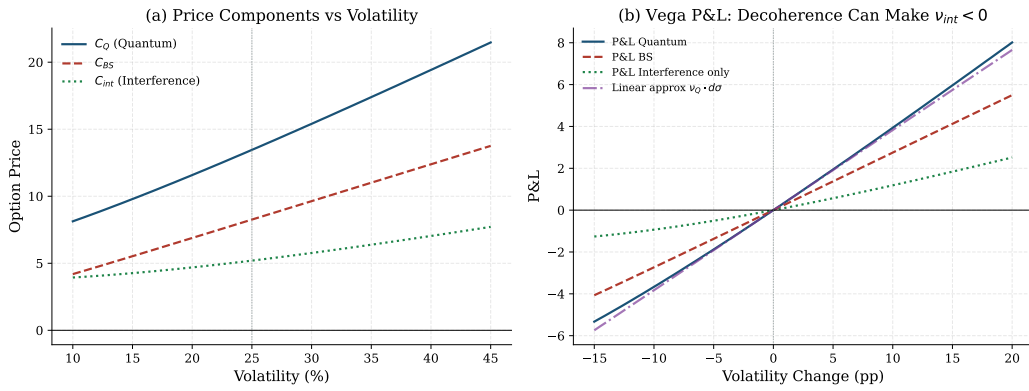


Figure 4: Vega analysis. (a) Price components as a function of volatility: the interference term C_{int} decreases at high σ due to decoherence. (b) Vega P&L: the quantum vega can be lower than the classical vega because increasing volatility destroys the phase coherence between the two signals, partially offsetting the classical spreading effect.

The interference theta contains a term proportional to $\Delta\omega$ (the frequency difference between the two signals), reflecting the fact that the interference pattern *moves* over time. As the interference fringes sweep across the strike price, the option can temporarily gain value even as time passes, i.e. a positive theta contribution that partially offsets the classical time decay. The total theta can even become positive in some windows, meaning that the option gains value as it approaches expiration, provided an interference fringe is moving through the in-the-money region.

Theta: Non-Monotone Time Decay from Phase Drift

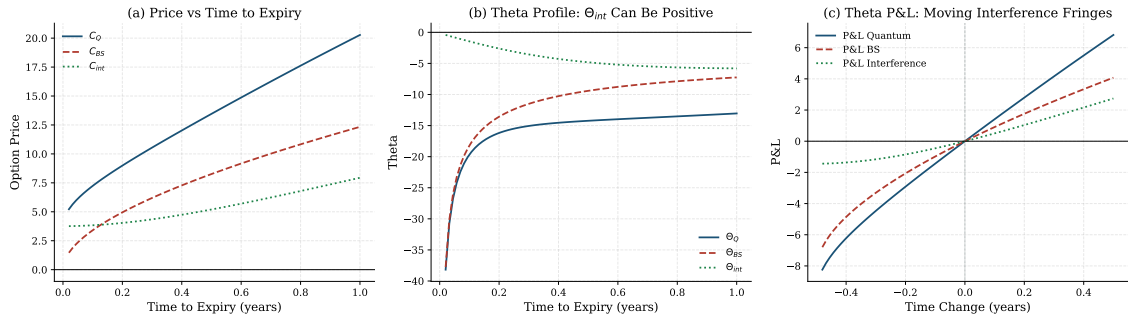


Figure 5: Theta analysis. (a) Option price versus time to expiry for the quantum and classical models. (b) The theta profile reveals that Θ_{int} can become positive at certain maturities, when a constructive interference fringe drifts through the in-the-money region. (c) Theta P&L decomposition, showing the oscillatory contribution from the moving interference pattern at phase velocity $v_\varphi = \Delta\omega/\Delta k$.

The **quantum rho** $\rho_Q = \partial C_Q/\partial r$ receives a similar interference correction. However, as in the classical case, the rho correction is typically small compared to the other Greeks and we do not dwell on it here.

6.2 The novel Greeks

Three sensitivities arise in our model that have no analogue in the Black–Scholes framework. They correspond to derivatives of the option price with respect to the three parameters that are specific to the wave-mechanical description.

The first is the **Planck Greek**, denoted Ξ and defined as

$$\Xi = \frac{\partial C_Q}{\partial \hbar_f} \quad (31)$$

This measures the sensitivity of the option price to the degree of market imperfection. When $\Xi > 0$, the option is worth more in an imperfect market than in a perfect one. The quantum effects enhance its value. When $\Xi < 0$, market imperfections reduce the option's value. For a practitioner, Ξ answers the question: “*how much am I paying (or being paid) for market microstructure?*” It provides a quantitative framework for assessing the impact of market quality on option prices.

Ξ (Planck Greek): How Market Imperfection Drives Option Value

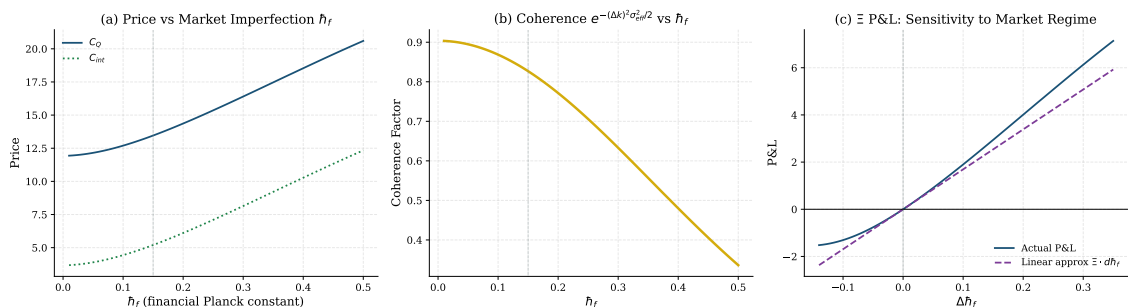


Figure 6: Planck Greek Ξ analysis. (a) The quantum price C_Q and the interference correction C_{int} as functions of the financial Planck constant \hbar_f . (b) The coherence factor $\exp[-(\Delta k)^2 \sigma_{eff}^2 / 2]$ decays as market imperfection grows, because σ_{eff} increases with \hbar_f . (c) P&L from a change in \hbar_f : the linear approximation Ξdh_f captures the local sensitivity but underestimates the non-linearity at large deviations.

The second novel Greek is the **phase Greek**, denoted Φ and defined as

$$\Phi = \frac{\partial C_Q}{\partial \varphi_0} = -2 w_1 w_2 e^{-rT} \int_{x_K}^{+\infty} |\psi_1| |\psi_2| \sin(\Delta\varphi) (S_0 e^x - K) dx \quad (32)$$

This measures the sensitivity of the option price to the *relative timing* of the two market signals. It is a purely wave-mechanical quantity. In a diffusive model, there is no phase and hence no phase sensitivity. A trader who knows that two announcements such as earnings and a central bank decision are coming, can use Φ to estimate how the *time gap* between the announcements affects the option price, even holding all other parameters fixed.

The third novel Greek is the **coherence Greek**, denoted κ and defined as

$$\kappa = \frac{\partial C_Q}{\partial (\Delta k)} = -(\Delta k) \sigma_{eff}^2 C_{int} \quad (33)$$

This measures the sensitivity of the option price to the *degree of disagreement* between the two market signals. The sign is negative when $C_{int} > 0$. Increasing disagreement between the signals destroys coherence and reduces the interference contribution. For a practitioner, κ answers the question: “*how much does the option price change if the market becomes more divided in its outlook?*”

Together, these three Greeks provide a trader with a complete picture of the wave-mechanical risks embedded in the option price, supplementing the classical Greeks that capture the diffusive risks.

Φ (Phase Greek): Option Price Sensitivity to Signal Timing

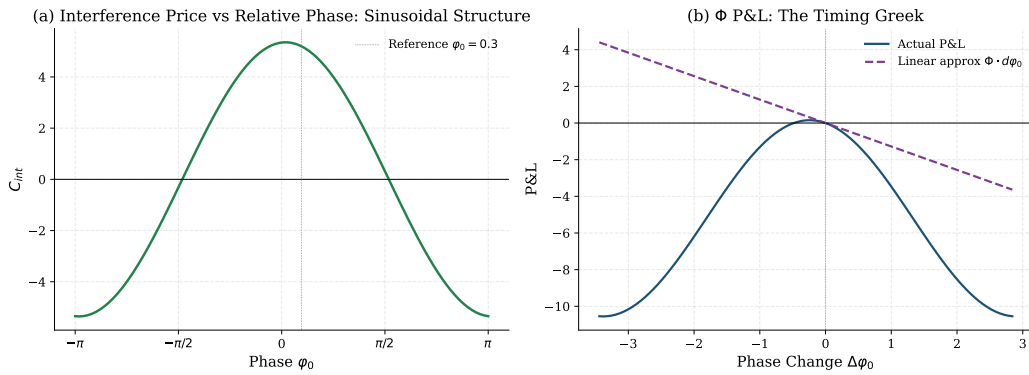


Figure 7: Phase Greek Φ analysis. (a) The interference price C_{int} traces a sinusoidal pattern as the relative phase φ_0 between the two market signals varies from $-\pi$ to $+\pi$. (b) P&L from a phase shift: the linear approximation $\Phi d\varphi_0$ is accurate near the reference phase but breaks down for large shifts, reflecting the inherently periodic structure.

κ (Coherence Greek): When Signals Diverge, Coherence Collapses

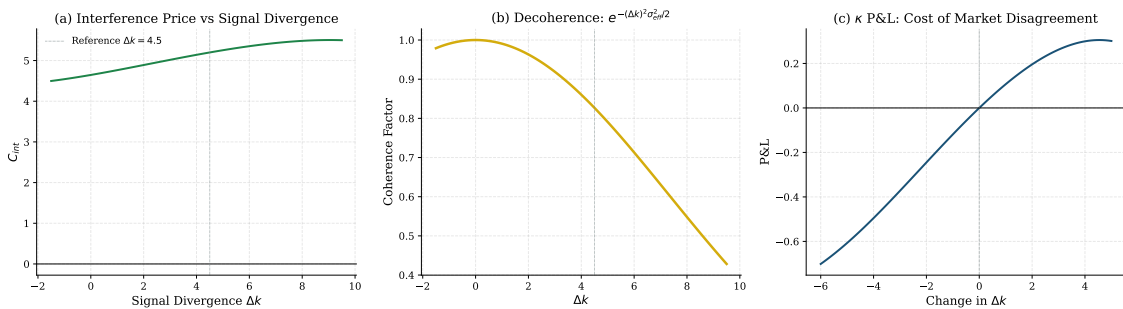


Figure 8: Coherence Greek κ analysis. (a) The interference price C_{int} as a function of the signal divergence Δk : the contribution peaks at moderate divergence and collapses at large Δk . (b) The coherence factor drops to zero as Δk increases, illustrating decoherence. (c) P&L from a change in signal divergence: increasing disagreement between market participants destroys interference value.

Complete Greeks Dashboard: Classical and Novel Sensitivities

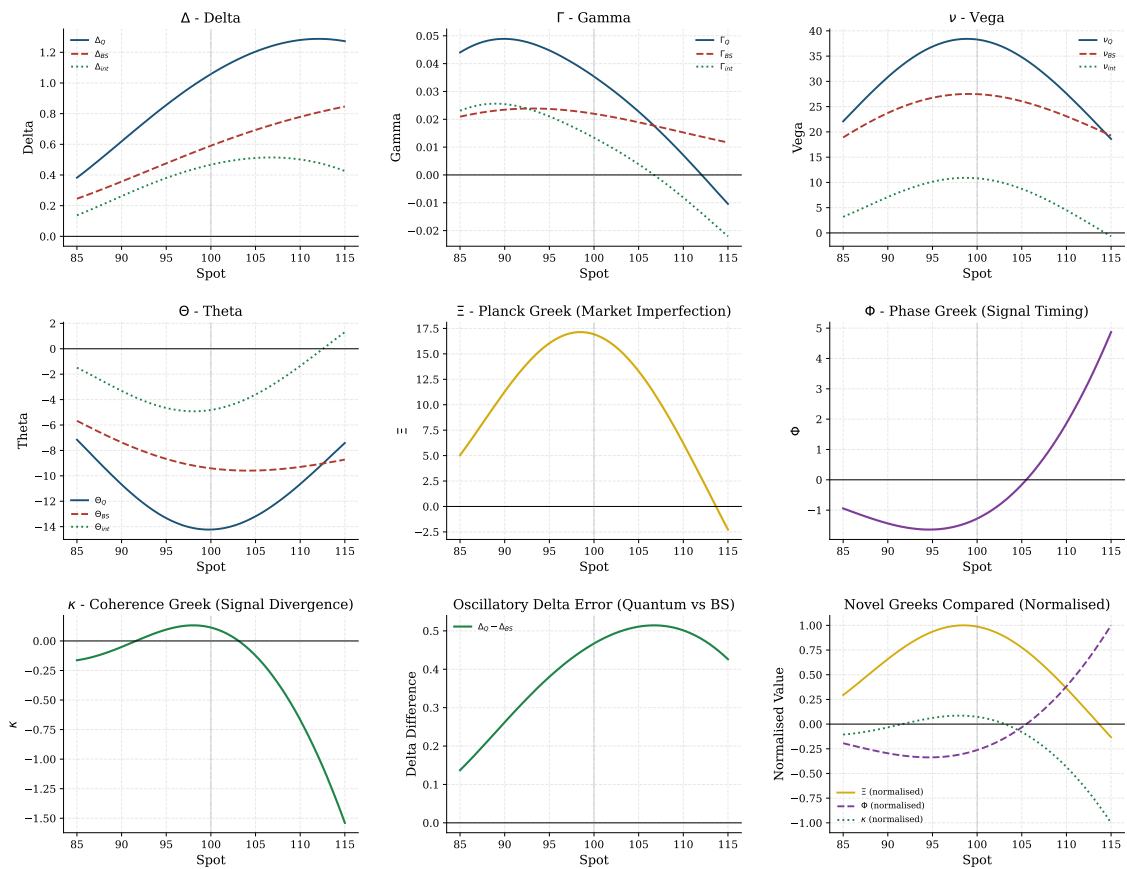


Figure 9: Complete Greeks dashboard as a function of the spot price. *Top row: the classical Greeks Δ , Γ , ν with their quantum and interference components. Middle row: Θ and the two novel Greeks Ξ (Planck) and Φ (phase). Bottom row: the coherence Greek κ , the oscillatory delta error $\Delta_Q - \Delta_{BS}$, and a normalised overlay of all three novel Greeks for comparison.*

6.3 Hedging strategy

The practical hedging of a quantum option requires holding Δ_Q shares of the underlying, rather than the classical Δ_{BS} shares. The hedging error from using the classical delta instead of the quantum delta is

$$\varepsilon = (\Delta_Q - \Delta_{BS}) \delta S \approx \Delta_{\text{int}} \delta S \quad (34)$$

where δS is the price move over the hedging interval. This error is not random noise. It is a *systematic*, oscillatory function of the spot price, determined by the interference parameters. A trader who estimates the interference parameters (from the observed smile for instance) can predict the sign and approximate magnitude of the hedging error and correct for it.

The Volatility Smile Emerges as an Interference Pattern

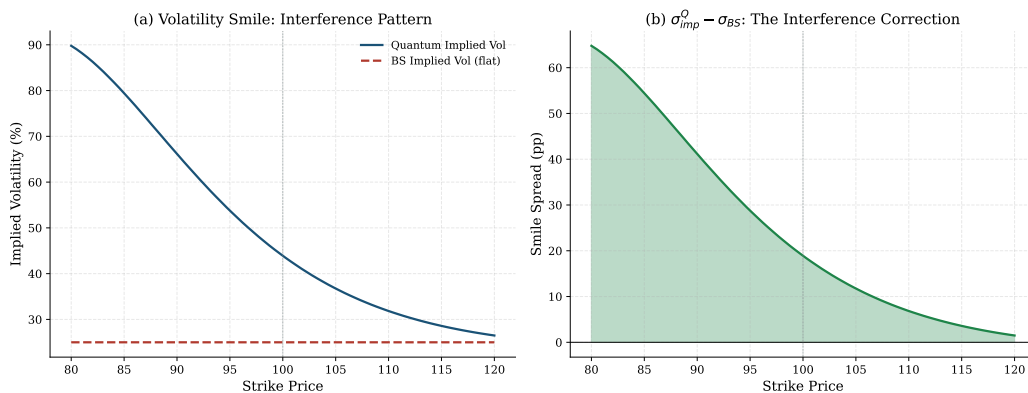


Figure 10: The volatility smile as an interference pattern. (a) *Implied volatility across strikes: the Black–Scholes implied volatility is flat at 25% by construction, while the quantum implied volatility exhibits a smile-like structure generated by the interference term.* (b) *The smile spread $\sigma_{\text{imp}}^Q - \sigma_{BS}$ in percentage points, showing the oscillatory correction whose amplitude is controlled by the coherence factor and whose frequency depends on Δk .*

Moreover, the gamma hedging strategy must account for the fact that $\Gamma_Q \neq \Gamma_{BS}$. In regions of price space where Γ_{int} is large and positive, the delta changes rapidly and the portfolio must be rebalanced more frequently. In regions where Γ_{int} is negative (partially cancelling the classical gamma), rebalancing can be less frequent. The optimal hedging frequency is therefore *state-dependent* in a way that the classical model does not predict.

7 The Classical Limit

Any generalisation of an established model must reduce to that model in the appropriate limit. We verify this for three distinct limiting procedures.

When $\hbar_f \rightarrow 0$, the effective width $\sigma_{\text{eff}} \rightarrow \sigma_0$, the wave numbers $k_j = p_j/\hbar_f \rightarrow \infty$ and the coherence factor $e^{-(\Delta k)^2 \sigma_{\text{eff}}^2/2} \rightarrow 0$ because $\Delta k \rightarrow \infty$. The interference term vanishes, and the quantum price reduces to the weighted average $C_Q \rightarrow w_1^2 C_1 + w_2^2 C_2$, i.e. a classical mixture of two Black–Scholes prices, with no wave effects. All novel Greeks vanish: $\Xi = \Phi = \kappa = 0$.

When $w_2 \rightarrow 0$ (a single market signal), the superposition reduces to $\psi = \psi_1$, the interference term vanishes identically and $C_Q \rightarrow C_1 = C_{BS}$. The model collapses exactly to the single-signal Black–Scholes world.

When $\Delta k \rightarrow 0$ (the two signals carry the same momentum, i.e. they agree completely), the cosine factor $\cos(\Delta\varphi) \rightarrow 1$ the coherence factor $\rightarrow 1$, and the interference term $C_{\text{int}} \rightarrow 2w_1 w_2 C_{BS}$, so

that $C_Q \rightarrow (w_1 + w_2)^2 C_{BS} = C_{BS}$. The two waves are identical, their superposition is a single wave of doubled amplitude and the classical price is recovered.

These three limits provide a reassuring consistency check. The quantum model is a genuine generalisation of Black–Scholes, not a replacement. It reduces to the classical framework whenever wave effects are negligible.

8 Conclusion

8.1 Contributions

This paper has proposed a wave-mechanical framework for option pricing that preserves the imaginary unit i as a structural element rather than rotating it away. Starting from a financial analogue of de Broglie’s hypothesis, we have derived a financial Schrödinger equation, identified four wave phenomena with clear market interpretations, obtained a closed-form pricing formula for European options and computed the full set of hedging Greeks, including three novel sensitivities that capture wave-specific risks.

The model offers several conceptual advantages. The volatility smile emerges naturally as an interference pattern, without requiring ad hoc modifications to the underlying process. Barrier option mispricing is explained by quantum tunnelling. False breakouts are a manifestation of diffraction. Post-announcement dynamics are described by wave-function collapse. Each of these explanations carries a strong physical intuition that can aid practitioners in developing qualitative understanding of market phenomena.

8.2 Calibration and empirical testing

The model introduces a small number of new parameters beyond Black–Scholes: the financial Planck constant \hbar_f , the wave number difference Δk , the weights w_1 and w_2 , and the initial phase φ_0 . In principle, these can be calibrated from the observed volatility surface. The coherence factor $e^{-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2}$ determines the amplitude of the smile oscillations, while the fringe spacing $2\pi/|\Delta k|$ determines their frequency in strike space. The phase φ_0 shifts the smile pattern horizontally. A systematic empirical study of this calibration, comparing the model’s performance against standard alternatives (Dupire, Heston, SABR) is the most important direction for future work.

An empirically attractive feature is that \hbar_f can be estimated as the gap between observed option prices and Black–Scholes prices. Markets known to be “more quantum” (less liquid, more informationally fragmented and more prone to microstructure effects) should exhibit larger values of \hbar_f .

8.3 Limitations and extensions

The present treatment is deliberately simplified in several respects. We have considered only two interfering signals, whereas real markets may contain many. The extension to N signals is straightforward in principle (the interference becomes a sum over all pairs) but increases the number of parameters. We have treated the potential $V(x, t)$ only qualitatively. A systematic classification of financial potentials (barrier, harmonic, step, periodic) and their effects on option pricing is a natural next step. We have not addressed American options, path-dependent options, or multi-asset options, each of which presents specific technical challenges (early exercise, history dependence, entanglement respectively) that deserve separate treatment.

Perhaps the most important conceptual limitation is the interpretation of $|\psi|^2$ as a risk-neutral density. In quantum mechanics, the Born rule ($|\psi|^2 = \text{probability density}$) is a postulate with deep experimental support. In finance, identifying $|\psi|^2$ with the risk-neutral density is an assumption that must be justified empirically. The classical limit analysis of Section 7 shows that this identification is at least consistent, but a deeper foundational discussion is warranted.

Despite these limitations, we believe that the wave-mechanical perspective opens a genuinely new direction in quantitative finance. The language of waves (interference, coherence, phase and tunnelling) provides an intuitive and mathematically rich framework for understanding market phenomena that lie beyond the reach of pure diffusion. Our hope is that this paper will encourage further exploration of this direction, both theoretical and empirical.

Paris-Saclay

9 Appendices

This section gathers the detailed calculations that support the results stated in the main text. The goal is to make each step explicit enough that the reader can follow (and reproduce) the algebra without having to fill in gaps. We try to emphasise the financial meaning of each intermediate result, not just the mathematical mechanics.

9.1 Derivation of the Financial Schrödinger Equation

The purpose of this section is to show that the financial Schrödinger equation is not an assumption but a *consequence* of two ingredients: the de Broglie plane wave and the classical energy-momentum relation. The logic mirrors exactly what Schrödinger did in 1926, transposed to our financial setting.

The financial plane wave

Our starting point is the financial plane wave postulated in Section 2 of the main text:

$$\psi(x, t) = A \exp[i(k_f x - \omega_f t)] \quad (35)$$

where $x = \ln(S/S_0)$ is the log-price, $k_f = p_f/\hbar_f$ is the financial wave number and $\omega_f = E_f/\hbar_f$ is the financial frequency. This wave encodes two pieces of information in its phase: the spatial oscillation rate k_f (related to the financial momentum p_f) and the temporal oscillation rate ω_f (related to the financial energy E_f).

The energy operator

We compute the time derivative of ψ :

$$\frac{\partial \psi}{\partial t} = -i \omega_f \psi = -i \frac{E_f}{\hbar_f} \psi \quad (36)$$

Multiplying both sides by $i\hbar_f$ gives

$$i\hbar_f \frac{\partial \psi}{\partial t} = E_f \psi \quad (37)$$

What this says is simple: the operator $i\hbar_f \partial/\partial t$, when applied to a plane wave, extracts its energy. The operator “reads” the temporal oscillation frequency and converts it into energy through the de Broglie relation $E_f = \hbar_f \omega_f$. In physics, this is the standard quantum energy operator. Here, it extracts the financial energy embedded in the oscillation of the market wave.

The momentum operator

We do the same thing with the spatial derivative. The first derivative gives

$$\frac{\partial \psi}{\partial x} = i k_f \psi = i \frac{p_f}{\hbar_f} \psi \quad (38)$$

so that

$$-i\hbar_f \frac{\partial \psi}{\partial x} = p_f \psi \quad (39)$$

The operator $-i\hbar_f \partial/\partial x$ extracts the financial momentum from the spatial oscillation pattern. A wave that oscillates rapidly across price space (large k_f) carries large momentum, and a slowly varying wave carries little.

Taking the second derivative:

$$\frac{\partial^2 \psi}{\partial x^2} = -k_f^2 \psi = -\frac{p_f^2}{\hbar_f^2} \psi \quad (40)$$

which we rearrange as

$$-\hbar_f^2 \frac{\partial^2 \psi}{\partial x^2} = p_f^2 \psi \quad (41)$$

Dividing by $2m_f$ (the financial mass):

$$-\frac{\hbar_f^2}{2m_f} \frac{\partial^2 \psi}{\partial x^2} = \frac{p_f^2}{2m_f} \psi \quad (42)$$

The right-hand side is the kinetic energy $p_f^2/(2m_f)$. The operator on the left extracts this kinetic energy from the *curvature* of the wave. A wave that bends sharply in price space (high k_f , rapid spatial oscillations) carries a lot of kinetic energy, whereas a flat wave carries none.

Assembling the equation

The classical energy–momentum relation for a particle of mass m_f in a potential $V(x, t)$ is

$$E_f = \frac{p_f^2}{2m_f} + V(x, t) \quad (43)$$

In our financial context, $m_f = 1/\sigma^2$, $p_f = \mu/\sigma^2$ and the kinetic term equals $\mu^2/(2\sigma^2)$. It measures the “cost” of sustaining a drift μ against a volatility σ : a large drift through a noisy environment requires a lot of kinetic energy.

Schrödinger’s key move was to replace every physical quantity in equation (43) by its corresponding operator, and then apply both sides to ψ . Using equations (37) and (42), this gives:

$$\underbrace{i\hbar_f \frac{\partial}{\partial t}}_{\text{energy operator}} \psi = \underbrace{-\frac{\hbar_f^2}{2m_f} \frac{\partial^2}{\partial x^2}}_{\text{kinetic energy operator}} \psi + V(x, t) \psi \quad (44)$$

Substituting $m_f = 1/\sigma^2$, so that $\hbar_f^2/(2m_f) = \hbar_f^2\sigma^2/2$, we obtain the **financial Schrödinger equation**:

$$\boxed{i\hbar_f \frac{\partial \psi}{\partial t} = -\frac{\hbar_f^2 \sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} + V(x, t) \psi} \quad (45)$$

Consistency check: the plane wave is a solution

As a sanity check, let us verify that the plane wave (35) satisfies equation (45) when $V = 0$. The left-hand side gives

$$i\hbar_f \cdot (-i\omega_f) \psi = \hbar_f \omega_f \psi = E_f \psi. \quad (46)$$

The right-hand side (with $V = 0$) gives

$$-\frac{\hbar_f^2 \sigma^2}{2} \cdot (-k_f^2) \psi = \frac{\hbar_f^2 \sigma^2 k_f^2}{2} \psi = \frac{(\hbar_f k_f)^2}{2m_f} \psi = \frac{p_f^2}{2m_f} \psi \quad (47)$$

Equating both sides requires $E_f = p_f^2/(2m_f)$, which is exactly the energy–momentum relation for a free particle ($V = 0$). The construction is self-consistent.

Why i is necessary and cannot be removed

It is worth pausing to understand why the imaginary unit i appears in equation (45) and why removing it changes the physics completely.

The plane wave oscillates in time as $e^{-i\omega_f t}$. Differentiating with respect to t brings down a factor $-i\omega_f$. We need to multiply by i to get $+\omega_f$ (and hence E_f/\hbar_f). If we were to remove i from the wave — replacing $e^{-i\omega_f t}$ by the real decaying exponential $e^{-\omega_f t}$ — the time derivative would give $-\omega_f$ directly, with no need for the i prefactor. The resulting equation would be a diffusion (heat) equation, not a wave equation.

In short, i and the oscillatory character of ψ are two sides of the same coin. The moment we decide that the financial wave oscillates rather than decays, i enters the equation necessarily. And it is this oscillation that produces interference, tunnelling and all the other wave phenomena exploited in the main text.

Recovery of Black–Scholes via Wick rotation

To make the connection to Black–Scholes explicit, consider the substitution $\hbar_f \rightarrow -i$ (equivalently, $t \rightarrow it$, the so-called Wick rotation). The left-hand side of (45) becomes

$$i \cdot (-i) \frac{\partial \psi}{\partial t} = + \frac{\partial \psi}{\partial t} \quad (48)$$

and the equation reduces to

$$\frac{\partial \psi}{\partial t} = \frac{\sigma^2}{2} \frac{\partial^2 \psi}{\partial x^2} - V(x, t) \psi \quad (49)$$

which is the heat equation with a source term — precisely the form of the Black–Scholes PDE after the standard change of variables. In this diffusive world, the propagator is a real, decaying Gaussian, the phase vanishes and with it all interference effects. Black–Scholes is the phase-less limit of our framework.

9.2 Derivation of the Quantum Financial Propagator

The propagator $K(x, T; x', 0)$ is the fundamental building block of the theory. It gives the probability amplitude for the log-price to go from x' at time 0 to x at time T , and every pricing formula ultimately reduces to an integral involving K . In this section, we derive it from scratch, discuss its properties in detail and propagate a Gaussian wave packet to obtain the effective width formula used in the pricing section.

Statement of the problem

We look for the Green's function of the free ($V = 0$) financial Schrödinger equation:

$$i\hbar_f \frac{\partial K}{\partial T} = -\frac{\hbar_f^2 \sigma^2}{2} \frac{\partial^2 K}{\partial x^2} \quad (50)$$

subject to the initial condition

$$K(x, 0; x', 0) = \delta(x - x') \quad (51)$$

The delta function initial condition means that at $T = 0$, the price is known with certainty to be at x' . The propagator describes how this perfect knowledge dissolves over time.

Solution by Fourier transform

We expand K in plane waves:

$$K(x, T; x', 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \hat{K}(k, T) e^{ik(x-x')} dk \quad (52)$$

The initial condition (51) requires $\hat{K}(k, 0) = 1$ for all k , which is just the Fourier representation of the Dirac delta.

Substituting (52) into (50), the second spatial derivative acts on e^{ikx} and brings down $-k^2$, while the time derivative acts only on \hat{K} . Each Fourier mode decouples, giving

$$i\hbar_f \frac{\partial \hat{K}}{\partial T} = \frac{\hbar_f^2 \sigma^2}{2} k^2 \hat{K} \quad (53)$$

This is a first-order ODE in T for each value of k . The fact that different k -modes evolve independently is the main reason why the Fourier approach works so well. It diagonalises the time evolution.

The solution with $\hat{K}(k, 0) = 1$ is

$$\hat{K}(k, T) = \exp\left(-\frac{i\hbar_f \sigma^2 k^2 T}{2}\right) \quad (54)$$

Let us take a moment to understand this result. Each mode k picks up a phase proportional to $k^2 T$. Modes with large $|k|$ (rapid spatial oscillations) rotate faster than modes with small $|k|$ (slow oscillations). This k -dependent phase velocity is the hallmark of *dispersion*. Different frequency components travel at different speeds, causing a wave packet to spread over time. In Black–Scholes (the heat equation), the corresponding factor is $\exp(-\sigma^2 k^2 T/2)$, which is real and decaying. High-frequency modes are *damped* rather than phase-shifted. In our equation, nothing is damped — everything rotates.

Inverse Fourier transform

The propagator in position space is obtained by plugging (54) back into (52):

$$K(x, T; x', 0) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} \exp\left(-\frac{i\hbar_f \sigma^2 k^2 T}{2} + ik(x - x')\right) dk \quad (55)$$

This is a Gaussian integral in k , of the form $\int \exp(-\alpha k^2 + \beta k) dk$ with

$$\alpha = \frac{i\hbar_f \sigma^2 T}{2}, \quad \beta = i(x - x') \quad (56)$$

The standard Gaussian integral formula, valid for $\text{Re}(\alpha) \geq 0$ (or by analytic continuation when α is purely imaginary), gives

$$\int_{-\infty}^{+\infty} \exp(-\alpha k^2 + \beta k) dk = \sqrt{\frac{\pi}{\alpha}} \exp\left(\frac{\beta^2}{4\alpha}\right) \quad (57)$$

We compute the exponent:

$$\frac{\beta^2}{4\alpha} = \frac{-(x - x')^2}{4 \cdot i\hbar_f \sigma^2 T/2} = \frac{-(x - x')^2}{2i\hbar_f \sigma^2 T} = \frac{i(x - x')^2}{2\hbar_f \sigma^2 T} = \frac{im_f(x - x')^2}{2\hbar_f T} \quad (58)$$

where we used $m_f = 1/\sigma^2$ and the identity $1/i = -i$ in the third step.

The prefactor gives

$$\sqrt{\frac{\pi}{\alpha}} = \sqrt{\frac{2\pi}{i\hbar_f \sigma^2 T}} = \sqrt{\frac{2\pi m_f}{i\hbar_f T}} \quad (59)$$

Dividing by 2π from the Fourier convention and collecting terms:

$$K(x, T; x', 0) = \sqrt{\frac{m_f}{2\pi i\hbar_f T}} \exp\left[\frac{im_f(x - x')^2}{2\hbar_f T}\right] \quad (60)$$

Physical and financial interpretation

This result deserves careful discussion, because the entire physics of the model is encoded in it.

The \sqrt{i} prefactor. The prefactor involves $1/\sqrt{i}$. Since $\sqrt{i} = e^{i\pi/4} = (1+i)/\sqrt{2}$, the propagator is complex from the outset, with real and imaginary parts of equal magnitude. There is no way to make it purely real without losing information. This is in contrast to the Black–Scholes (heat) propagator, $K_{\text{BS}} = (2\pi\sigma^2T)^{-1/2} \exp[-(x-x')^2/(2\sigma^2T)]$, which is real and positive everywhere.

The oscillating exponential. The exponential $\exp[im_f(x-x')^2/(2\hbar_fT)]$ has modulus 1 for all values of $x-x'$, since its argument is purely imaginary. Unlike the Gaussian kernel of Black–Scholes, which decays exponentially for large displacements, the quantum propagator has constant magnitude across all of price space. In principle, every price level is “reachable” with equal amplitude.

This might seem unphysical at first, but it is not. When we integrate K against a localised initial state (a wave packet of finite width σ_0), the rapid oscillations of the phase for large $|x-x'|$ cancel out through destructive interference, and the result is effectively localised. The key point is that this localisation is achieved *dynamically*, through phase cancellation, rather than being built into the propagator from the start. This is precisely why tunnelling is possible. The propagator does not vanish behind barriers, it just oscillates rapidly there, and in certain configurations these oscillations fail to cancel completely, allowing a non-zero transmitted amplitude.

The quadratic phase. The phase of the propagator grows as $(x-x')^2/T$. It is quadratic in the displacement. For a fixed time T , distant points carry a rapidly varying phase. When we integrate against a smooth function, contributions from far-away points wash out (the oscillations average to zero), and only points near the *stationary phase point* survive. This is the semi-classical approximation and it reproduces Black–Scholes. The corrections beyond this approximation — the contributions from points that are not quite at the stationary phase — are the quantum effects of our model.

The $\hbar_f \rightarrow 0$ limit. When $\hbar_f \rightarrow 0$, the phase $m_f(x-x')^2/(2\hbar_fT)$ oscillates infinitely fast. In this limit, any integral of K against a smooth function is dominated by the single point where the phase is stationary, namely $x=x'$ (or $x=x'+\mu T$ if there is a drift). This is the classical trajectory. The price goes exactly where the drift takes it, with no uncertainty. Quantum corrections — tunnelling, interference — all vanish. This is the formal sense in which $\hbar_f \rightarrow 0$ recovers the classical limit.

Side-by-side comparison with Black–Scholes

For easy reference, here are both propagators for a free particle ($V=0$) with financial mass $m_f=1/\sigma^2$:

Quantum financial propagator:

$$K_{\text{Q}} = \sqrt{\frac{m_f}{2\pi i \hbar_f T}} \exp\left[\frac{im_f(x-x')^2}{2\hbar_f T}\right] \quad (61)$$

Black–Scholes (heat) propagator:

$$K_{\text{BS}} = \sqrt{\frac{m_f}{2\pi T}} \exp\left[-\frac{m_f(x-x')^2}{2T}\right] \quad (62)$$

We pass from K_{Q} to K_{BS} by the substitution $\hbar_f \rightarrow -i$ (Wick rotation). The quantum propagator oscillates and has constant modulus, whereas the classical one decays and is purely real. This single difference is responsible for all the wave phenomena discussed in the main text.

Propagation of a Gaussian wave packet

To prepare the ground for the pricing calculation, we now propagate an initial Gaussian wave packet through the free propagator. The initial state, centred at $x_0 = 0$ with width σ_0 and momentum p_0 is

$$\psi_0(x) = \left(\frac{1}{2\pi\sigma_0^2} \right)^{1/4} \exp\left(-\frac{x^2}{4\sigma_0^2} + \frac{ip_0x}{\hbar_f} \right) \quad (63)$$

The wave function at time T is obtained by convolution:

$$\psi(x, T) = \int_{-\infty}^{+\infty} K(x, T; x', 0) \psi_0(x') dx' \quad (64)$$

The integrand involves the product of two Gaussians in x' (one from K and one from ψ_0), so the integral is itself Gaussian. Collecting the terms in x' from the exponents:

$$\text{exponent in } x' = -\frac{x'^2}{4\sigma_0^2} + \frac{ip_0x'}{\hbar_f} + \frac{im_f}{2\hbar_f T} (x^2 - 2xx' + x'^2) \quad (65)$$

Grouping the x'^2 terms:

$$x'^2 \text{ coefficient : } -\frac{1}{4\sigma_0^2} + \frac{im_f}{2\hbar_f T} = -\frac{1}{4\sigma_0^2} \left(1 - \frac{2im_f\sigma_0^2}{\hbar_f T} \right) \equiv -\frac{1}{4\tilde{\sigma}^2} \quad (66)$$

where we defined the *complex width*

$$\tilde{\sigma}^2(T) = \frac{\sigma_0^2}{1 - 2im_f\sigma_0^2/(\hbar_f T)} = \sigma_0^2 + \frac{i\hbar_f\sigma^2 T}{2} \quad (67)$$

The x' -linear terms combine into a coefficient β that depends on x and p_0 . Completing the square and evaluating the Gaussian integral, we obtain (after some algebra):

$$\psi(x, T) = \left(\frac{1}{2\pi\sigma_{\text{eff}}^2} \right)^{1/4} \frac{1}{\sqrt{1 + i\hbar_f T/(2m_f\sigma_0^2)}} \exp\left[-\frac{(x - \mu T)^2}{4\tilde{\sigma}^2(T)} + i\phi(x, T) \right] \quad (68)$$

where $\mu = p_0\sigma^2$ is the drift implied by the initial momentum, and $\phi(x, T)$ is a global phase that we do not write out in full (it does not affect the probability density $|\psi|^2$).

The key result is the *modulus*:

$$|\psi(x, T)| = \left(\frac{1}{2\pi\sigma_{\text{eff}}^2} \right)^{1/4} \exp\left[-\frac{(x - \mu T)^2}{4\sigma_{\text{eff}}^2} \right] \quad (69)$$

which is a Gaussian centred at μT (the wave packet follows the classical drift) with width

$$\boxed{\sigma_{\text{eff}}^2(T) = \sigma_0^2 + \frac{\hbar_f^2 T^2 \sigma^2}{4\sigma_0^2}} \quad (70)$$

This formula has a clean interpretation. The first term σ_0^2 is the initial uncertainty. The second term is the *quantum spreading*. It grows as T^2 (not T as in classical diffusion), it is proportional to \hbar_f^2 (so it vanishes in the classical limit) and it is inversely proportional to σ_0^2 . This last feature is a manifestation of the financial uncertainty principle. A wave packet that is initially very narrow in price space (σ_0 small) must contain a broad range of momentum components, and these components disperse rapidly, causing the packet to spread faster.

The crossover between the initial width dominated regime and the quantum spreading dominated regime occurs at the *coherence time*

$$T_{\text{coh}} = \frac{2\sigma_0^2}{\hbar_f \sigma} \quad (71)$$

For $T \ll T_{\text{coh}}$, the quantum model and Black–Scholes give similar results. For $T \gg T_{\text{coh}}$, the quantum spreading dominates and the two models diverge. Calibrating \hbar_f from market data therefore tells us the time horizon beyond which wave effects become relevant.

9.3 Explicit Computation of the Interference Greeks

In this subsection, we derive closed-form expressions for the hedging sensitivities (Greeks) associated with the interference term C_{int} . Since the total quantum price is $C_Q = w_1^2 C_1 + w_2^2 C_2 + C_{\text{int}}$ and the Greeks of C_1 and C_2 are the standard Black–Scholes Greeks (well documented), we focus exclusively on the derivatives of C_{int} .

Notation

To keep expressions manageable, we introduce the following shorthand. The coherence factor is $\mathcal{C} = \exp[-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2]$. The overall prefactor is $\mathcal{G} = 2 w_1 w_2 e^{-rT} \mathcal{C}$. The two modified Black–Scholes terms are

$$\mathcal{F}_1 = S_0 e^{\bar{\mu}T} \text{Re}\left(e^{i\Delta\varphi_K} \mathcal{N}(d_1^{\text{int}})\right) \quad (72)$$

$$\mathcal{F}_2 = K \text{Re}\left(e^{i\Delta\varphi_K} \mathcal{N}(d_2^{\text{int}})\right) \quad (73)$$

so that $C_{\text{int}} = \mathcal{G} [\mathcal{F}_1 - \mathcal{F}_2]$. We also write $\Delta\varphi_K = \Delta k \cdot x_K - \Delta\omega T + \varphi_0$ for the interference phase at the strike, where $x_K = \ln(K/S_0)$.

Interference delta: Δ_{int}

The delta is $\partial C_{\text{int}} / \partial S_0$. The spot price S_0 enters C_{int} through three channels: the explicit factor S_0 in \mathcal{F}_1 , the log-moneyness $x_K = \ln(K/S_0)$ which affects both the d -variables and the interference phase, and nothing else since the coherence factor \mathcal{C} does not depend on S_0 . The relevant partial derivative is $\partial x_K / \partial S_0 = -1/S_0$.

The interference phase depends on S_0 through x_K :

$$\frac{\partial \Delta\varphi_K}{\partial S_0} = \Delta k \frac{\partial x_K}{\partial S_0} = -\frac{\Delta k}{S_0} \quad (74)$$

For the $\mathcal{N}(d_j^{\text{int}})$ terms, we recall that $d_j^{\text{int}} = (-x_K + \bar{\mu}T + c_j + i\Delta k \sigma_{\text{eff}}^2) / \sigma_{\text{eff}}$ where $c_1 = \sigma_{\text{eff}}^2$ and $c_2 = 0$. The derivative with respect to x_K is $\partial d_j^{\text{int}} / \partial x_K = -1/\sigma_{\text{eff}}$, so

$$\frac{\partial \mathcal{N}(d_j^{\text{int}})}{\partial S_0} = -\frac{n(d_j^{\text{int}})}{\sigma_{\text{eff}}} \cdot \frac{\partial x_K}{\partial S_0} = \frac{n(d_j^{\text{int}})}{S_0 \sigma_{\text{eff}}} \quad (75)$$

where $n(z) = (2\pi)^{-1/2} \exp(-z^2/2)$ is the standard normal density, extended to complex arguments by analytic continuation.

Finally, the derivative of $e^{i\Delta\varphi_K}$ with respect to S_0 brings down a factor $i\partial\Delta\varphi_K / \partial S_0 = -i\Delta k / S_0$.

Assembling all contributions and being careful to take the real part only at the end, since Re does

not commute with differentiation of the complex exponential, the interference delta is

$$\begin{aligned} \Delta_{\text{int}} = & \frac{\mathcal{G}}{S_0} \left[e^{\bar{\mu}T} \operatorname{Re} \left(e^{i\Delta\varphi_K} \mathcal{N}(d_1^{\text{int}}) \right) \right. \\ & + \Delta k \operatorname{Re} \left(i e^{i\Delta\varphi_K} \left[S_0 e^{\bar{\mu}T} \mathcal{N}(d_1^{\text{int}}) - K \mathcal{N}(d_2^{\text{int}}) \right] \right) \\ & \left. + \frac{1}{\sigma_{\text{eff}}} \operatorname{Re} \left(e^{i\Delta\varphi_K} \left[S_0 e^{\bar{\mu}T} n(d_1^{\text{int}}) - K n(d_2^{\text{int}}) \right] \right) \right] \end{aligned} \quad (76)$$

The three lines have distinct financial meanings. The first line is the “direct” delta: changing S_0 changes the intrinsic value through the S_0 prefactor in \mathcal{F}_1 . The second line, proportional to Δk , is the **phase delta**: changing S_0 shifts the strike’s position within the interference pattern. This is the term that makes the delta oscillatory. The factor $\operatorname{Re}(i e^{i\Delta\varphi_K} \dots) = -\operatorname{Im}(e^{i\Delta\varphi_K} \dots)$ introduces a sine where the classical delta has only monotone functions. The third line, proportional to $1/\sigma_{\text{eff}}$, comes from the variation of the cumulative normal \mathcal{N} with the strike. It is the interference analogue of the $n(d_1)$ term in the Black–Scholes delta.

The amplitude of the delta oscillations is governed by $\Delta k \cdot \mathcal{G}$. When Δk is small (the two signals agree), the oscillations are weak and $\Delta_Q \approx \Delta_{\text{BS}}$. When Δk is large, the oscillations are pronounced but damped by the coherence factor \mathcal{C} , which prevents the delta from blowing up.

Interference gamma: Γ_{int}

The gamma is the second derivative of the price with respect to S_0 , or equivalently the derivative of the delta. Rather than differentiating the already lengthy expression (76), it is more transparent to work in log-moneyness coordinates and use the relation

$$\Gamma = \frac{1}{S_0^2} \left(\frac{\partial^2 C}{\partial x_K^2} - \frac{\partial C}{\partial x_K} \right) \quad (77)$$

When we differentiate C_{int} twice with respect to x_K , the dominant contribution comes from the second derivative of $\cos(\Delta\varphi_K)$, which produces $-(\Delta k)^2 \cos(\Delta\varphi_K)$, plus a subdominant term from the second derivative of $\mathcal{N}(d_j^{\text{int}})$ through the chain rule. The leading-order result is

$$\Gamma_{\text{int}} \approx -\frac{(\Delta k)^2}{S_0^2} C_{\text{int}} + \frac{\mathcal{G}}{S_0^2 \sigma_{\text{eff}}} \operatorname{Re} \left(e^{i\Delta\varphi_K} \left[S_0 e^{\bar{\mu}T} n'_1 - K n'_2 \right] \right) \quad (78)$$

where $n'_j = -d_j^{\text{int}} n(d_j^{\text{int}})/\sigma_{\text{eff}}$.

The first term is the most striking. The interference gamma is approximately *proportional to the interference price itself*, with a factor $-(\Delta k)^2/S_0^2$. Where $C_{\text{int}} > 0$ (constructive interference enhances the price), the gamma correction is negative (the price curvature is reduced). Where $C_{\text{int}} < 0$ (destructive interference), the gamma correction is positive. The magnitude scales as $(\Delta k)^2$: the more the two signals disagree, the larger the gamma correction. This confirms the financial intuition that conflicting information amplifies convexity risk.

Interference vega: ν_{int}

The vega is $\partial C_{\text{int}}/\partial\sigma$. Volatility enters C_{int} through two main channels: the effective width $\sigma_{\text{eff}}^2 = \sigma_0^2 + \hbar_f^2 \sigma^4 T^2 / (4\sigma_0^2)$ and the drifts $\mu_j = \hbar_f k_j \sigma^2$. The most important channel is the coherence factor $\mathcal{C} = \exp[-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2]$ because it controls the overall strength of the interference.

The derivative of \mathcal{C} with respect to σ is

$$\frac{\partial \mathcal{C}}{\partial \sigma} = -(\Delta k)^2 \sigma_{\text{eff}} \frac{\partial \sigma_{\text{eff}}}{\partial \sigma} \mathcal{C} \quad (79)$$

where

$$\frac{\partial \sigma_{\text{eff}}}{\partial \sigma} = \frac{\hbar_f^2 \sigma^3 T^2}{2\sigma_0^2 \sigma_{\text{eff}}} \quad (80)$$

This gives the dominant (“decoherence”) contribution to the vega:

$$\nu_{\text{int}}^{(\text{decoh})} = \frac{\partial \mathcal{G}}{\partial \sigma} \cdot [\mathcal{F}_1 - \mathcal{F}_2] = -(\Delta k)^2 \frac{\hbar_f^2 \sigma^3 T^2}{2\sigma_0^2} C_{\text{int}} \quad (81)$$

This contribution is *negative* when $C_{\text{int}} > 0$. Increasing σ makes the effective width σ_{eff} larger, which strengthens the decoherence factor and washes out the interference pattern. Physically, higher volatility “scrambles” the phase relationship between the two signals, like thermal noise destroying the coherence of a light beam.

There is also a positive contribution from the variation of \mathcal{F}_1 and \mathcal{F}_2 with σ_{eff} through the d_j^{int} variables. This is the usual “spreading” effect. Higher volatility broadens the distribution and tends to increase option value. The total interference vega is

$$\boxed{\nu_{\text{int}} = \underbrace{-(\Delta k)^2 \frac{\hbar_f^2 \sigma^3 T^2}{2\sigma_0^2} C_{\text{int}}}_{\text{decoherence effect } (<0)} + \underbrace{\mathcal{G} \frac{\partial(\mathcal{F}_1 - \mathcal{F}_2)}{\partial \sigma}}_{\text{spreading effect } (>0)}} \quad (82)$$

The total quantum vega $\nu_Q = w_1^2 \nu_1 + w_2^2 \nu_2 + \nu_{\text{int}}$ can become negative when Δk is large enough for the decoherence effect to overwhelm both the spreading effect and the classical vega contributions. This corresponds to a regime where the two market signals are so contradictory that increasing volatility destroys more value (through decoherence) than it creates (through spreading). As far as we know, this is a prediction unique to the wave-mechanical framework.

Interference theta: Θ_{int}

The theta is $\partial C_{\text{int}} / \partial T$. Time enters C_{int} through the discount factor e^{-rT} , the effective width $\sigma_{\text{eff}}(T)$, the drift terms $\bar{\mu}T$, the coherence factor and, most importantly, the interference phase via $-\Delta\omega T$.

The most distinctive contribution comes from the time dependence of the interference pattern itself. Differentiating $\cos(\Delta k x_K - \Delta\omega T + \varphi_0)$ with respect to T gives $\Delta\omega \sin(\Delta k x_K - \Delta\omega T + \varphi_0)$. The associated theta contribution is

$$\Theta_{\text{int}}^{(\text{phase})} = \Delta\omega \mathcal{G} \text{Im} \left(e^{i\Delta\varphi_K} \left[S_0 e^{\bar{\mu}T} \mathcal{N}(d_1^{\text{int}}) - K \mathcal{N}(d_2^{\text{int}}) \right] \right) \quad (83)$$

This term is oscillatory in T . As time passes, the interference fringes *move* through price space at phase velocity $v_\varphi = \Delta\omega / \Delta k$. There are moments when a constructive fringe passes through the in-the-money region, temporarily boosting the probability that the option expires in the money. At those moments, $\Theta_{\text{int}}^{(\text{phase})} > 0$, and if the effect is strong enough, the total theta Θ_Q becomes positive — the option *gains* value as it approaches expiration, against the usual time-decay intuition.

The full interference theta collects four contributions:

$$\boxed{\Theta_{\text{int}} = \underbrace{-r C_{\text{int}}}_{\text{discounting}} + \underbrace{\Theta_{\text{int}}^{(\text{phase})}}_{\text{fringe drift}} + \underbrace{\Theta_{\text{int}}^{(\text{decoh})}}_{\text{decoherence growth}} + \underbrace{\Theta_{\text{int}}^{(\text{drift})}}_{\text{drift accumulation}}} \quad (84)$$

The decoherence term $\Theta_{\text{int}}^{(\text{decoh})} < 0$ because σ_{eff} grows with T , steadily washing out the interference. The drift term depends on the sign of $\bar{\mu}$.

The Planck Greek: Ξ

The sensitivity to the fundamental quantum parameter \hbar_f is

$$\Xi = \frac{\partial C_Q}{\partial \hbar_f} \quad (85)$$

Only C_{int} depends on \hbar_f at leading order. The individual prices C_1 and C_2 depend on σ_{eff} , but this is a higher-order effect. The main channel is the coherence factor, through σ_{eff}^2 :

$$\frac{\partial \sigma_{\text{eff}}^2}{\partial \hbar_f} = \frac{\hbar_f \sigma^4 T^2}{2\sigma_0^2}, \quad (86)$$

which gives

$$\frac{\partial C}{\partial \hbar_f} = -\frac{(\Delta k)^2}{2} \cdot \frac{\hbar_f \sigma^4 T^2}{2\sigma_0^2} C \quad (87)$$

There is also a contribution from the wave numbers $k_j = p_j/\hbar_f$. Since the physical momentum p_j is fixed, changing \hbar_f changes $\Delta k = \Delta p/\hbar_f$ and therefore shifts the entire interference pattern. The complete expression is

$$\Xi = C_{\text{int}} \left[-\frac{(\Delta k)^2 \hbar_f \sigma^4 T^2}{4\sigma_0^2} + \frac{2 \Delta k}{\hbar_f} \frac{\partial \Delta \varphi_K}{\partial (\Delta k)} \right] + \mathcal{G} \frac{\partial (\mathcal{F}_1 - \mathcal{F}_2)}{\partial \hbar_f} \quad (88)$$

In the limit $\hbar_f \rightarrow 0$, $\Xi \rightarrow 0$: there is no quantum sensitivity in a perfectly classical market. This is consistent with the classical limit of the model.

The phase Greek: Φ

The phase φ_0 enters C_{int} only through $\Delta \varphi_K = \Delta k \cdot x_K - \Delta \omega T + \varphi_0$. The derivative is clean:

$$\frac{\partial}{\partial \varphi_0} \text{Re} \left(e^{i\Delta \varphi_K} f \right) = -\text{Im} \left(e^{i\Delta \varphi_K} f \right) \quad (89)$$

for any complex quantity f . Therefore:

$$\Phi = -\mathcal{G} \left[S_0 e^{\bar{\mu}T} \text{Im} \left(e^{i\Delta \varphi_K} \mathcal{N}(d_1^{\text{int}}) \right) - K \text{Im} \left(e^{i\Delta \varphi_K} \mathcal{N}(d_2^{\text{int}}) \right) \right] \quad (90)$$

This is exactly C_{int} with Re replaced by $-\text{Im}$: the phase Greek is the “quadrature complement” of the interference price. The relationship between the two can be expressed compactly by noting that

$$C_{\text{int}} + i\Phi = \mathcal{G} \left[S_0 e^{\bar{\mu}T} e^{i\Delta \varphi_K} \mathcal{N}(d_1^{\text{int}}) - K e^{i\Delta \varphi_K} \mathcal{N}(d_2^{\text{int}}) \right] \quad (91)$$

so that C_{int} and Φ are the real and (minus the) imaginary parts of the same complex object. This implies the constraint

$$C_{\text{int}}^2 + \Phi^2 = \mathcal{G}^2 \left| S_0 e^{\bar{\mu}T} \mathcal{N}(d_1^{\text{int}}) - K \mathcal{N}(d_2^{\text{int}}) \right|^2 \quad (92)$$

The right hand side is positive and independent of φ_0 . So as the phase φ_0 varies, the point (C_{int}, Φ) traces a circle of fixed radius. The total “interference risk” — combining both the price impact and the timing sensitivity — is constant. Only its decomposition changes. When C_{int} is maximal (constructive interference), $\Phi = 0$ (the price is at a phase extremum, insensitive to small timing shifts). When $C_{\text{int}} = 0$ (we sit exactly on an interference node), $|\Phi|$ is maximal (the slightest change in timing tips the balance one way or the other). This is a useful insight for risk management. The timing risk is largest precisely when the interference contribution to the price is near zero.

The coherence Greek: κ

The derivative with respect to Δk involves both the coherence factor and the phase:

$$\frac{\partial \mathcal{C}}{\partial(\Delta k)} = -(\Delta k) \sigma_{\text{eff}}^2 \mathcal{C}, \quad \frac{\partial \Delta \varphi_K}{\partial(\Delta k)} = x_K \quad (93)$$

The complete expression is

$$\boxed{\kappa = -(\Delta k) \sigma_{\text{eff}}^2 C_{\text{int}} + x_K \Phi} \quad (94)$$

where Φ is the phase Greek from equation (90).

The first term captures the decoherence effect. Increasing the disagreement between signals (larger Δk) weakens the coherence and reduces the interference contribution. The second term, proportional to $x_K = \ln(K/S_0)$, accounts for the fact that changing Δk shifts the fringe positions relative to the strike.

For an at-the-money option ($x_K = 0$, i.e. $K = S_0$), the second term vanishes and the coherence Greek simplifies to

$$\kappa|_{\text{ATM}} = -(\Delta k) \sigma_{\text{eff}}^2 C_{\text{int}}|_{\text{ATM}} \quad (95)$$

This is a pleasingly simple result: for ATM options, the exposure to signal divergence is just proportional to the interference price, with a known coefficient. It gives a quick way to assess how much the option price depends on the degree of market consensus.

Consistency relations

The interference Greeks are not all independent. They satisfy identities that serve as useful cross-checks.

The first relation, already established, links C_{int} and Φ . They are the real and imaginary parts of a complex price, constrained to lie on a circle (equation (92)).

The second relation links κ to both C_{int} and Φ via equation (94). For ATM options, κ depends only on C_{int} . For away-from-the-money options, Φ enters with a weight x_K .

A third relation connects the phase theta $\Theta_{\text{int}}^{(\text{phase})}$ (equation (83)) to Φ . Comparing the two expressions, we find

$$\Theta_{\text{int}}^{(\text{phase})} = -\Delta \omega \Phi \quad (96)$$

which says that the time drift of the interference pattern is proportional to the phase sensitivity, scaled by the frequency difference. This makes sense: Φ measures how much the price changes per unit shift in phase, and $\Delta \omega$ is the rate at which the phase naturally shifts over time. Their product gives the rate of price change due to the moving fringes.

These relations can be used in practice as sanity checks when implementing the model numerically. If the computed Greeks do not satisfy these identities (within numerical precision), there is a bug.

9.4 Gaussian Integration of the Interference Term

We now provide the full derivation of the closed-form expression for the interference contribution to the option price, equation (27) of the main text.

Wave packets at maturity

Each component ψ_j ($j = 1, 2$) evolves freely under the financial Schrödinger equation. Starting from the Gaussian initial state (63) with momentum p_j , the result at time T was derived in

section 9.2. The modulus of each wave packet is

$$|\psi_j(x, T)| = \left(\frac{1}{2\pi\sigma_{\text{eff}}^2} \right)^{1/4} \exp \left[-\frac{(x - \mu_j T)^2}{4\sigma_{\text{eff}}^2} \right] \quad (97)$$

where $\mu_j = \hbar_f k_j \sigma^2$ and σ_{eff} is given by (70). Both packets share the same width σ_{eff} (since they start with the same σ_0) but have different centres $\mu_j T$.

Product of moduli

The interference term (equation (27) of the main text) involves the product $|\psi_1(x, T)| \cdot |\psi_2(x, T)|$. Using the standard identity for the product of two Gaussians with the same variance but different means,

$$\exp \left[-\frac{(x - a)^2}{4s^2} \right] \cdot \exp \left[-\frac{(x - b)^2}{4s^2} \right] = \exp \left[-\frac{(a - b)^2}{8s^2} \right] \cdot \exp \left[-\frac{(x - \bar{c})^2}{2s^2} \right] \quad (98)$$

with $\bar{c} = (a + b)/2$, we get

$$|\psi_1| |\psi_2| = \frac{1}{\sqrt{2\pi\sigma_{\text{eff}}^2}} \exp \left[-\frac{(\Delta\mu T)^2}{8\sigma_{\text{eff}}^2} \right] \exp \left[-\frac{(x - \bar{\mu}T)^2}{2\sigma_{\text{eff}}^2} \right] \quad (99)$$

where $\bar{\mu} = (\mu_1 + \mu_2)/2$ and $\Delta\mu = \mu_1 - \mu_2$. The first exponential is a constant (in x) that penalises the separation between the two wave packets: the further apart their centres, the smaller their overlap.

Reduction to standard form

The integral we need to evaluate is

$$C_{\text{int}} = 2w_1 w_2 e^{-rT} \int_{x_K}^{+\infty} |\psi_1| |\psi_2| \cos(\Delta k \cdot x - \Delta\omega T + \varphi_0) (S_0 e^x - K) dx \quad (100)$$

Substituting (99) and splitting the payoff into two terms, this becomes

$$C_{\text{int}} = 2w_1 w_2 e^{-rT} \mathcal{P} [S_0 I_1 - K I_2] \quad (101)$$

where $\mathcal{P} = (2\pi\sigma_{\text{eff}}^2)^{-1/2} \exp[-(\Delta\mu T)^2/(8\sigma_{\text{eff}}^2)]$ collects the overlap prefactor, and

$$I_1 = \int_{x_K}^{+\infty} \exp \left[-\frac{(x - \bar{\mu}T)^2}{2\sigma_{\text{eff}}^2} + x \right] \cos(\Delta k \cdot x - \Delta\omega T + \varphi_0) dx \quad (102)$$

$$I_2 = \int_{x_K}^{+\infty} \exp \left[-\frac{(x - \bar{\mu}T)^2}{2\sigma_{\text{eff}}^2} \right] \cos(\Delta k \cdot x - \Delta\omega T + \varphi_0) dx \quad (103)$$

Evaluation of I_2

The key step is to convert the cosine to complex exponential form:

$$\cos(\Delta\varphi(x)) = \text{Re} \left(e^{i(\Delta k \cdot x - \Delta\omega T + \varphi_0)} \right) \quad (104)$$

So $I_2 = \text{Re}(J_2)$ where

$$J_2 = e^{i(-\Delta\omega T + \varphi_0)} \int_{x_K}^{+\infty} \exp \left[-\frac{(x - \bar{\mu}T)^2}{2\sigma_{\text{eff}}^2} + i\Delta k x \right] dx \quad (105)$$

We substitute $u = x - \bar{\mu}T$ and complete the square in the exponent:

$$\begin{aligned} -\frac{u^2}{2\sigma_{\text{eff}}^2} + i\Delta k(u + \bar{\mu}T) &= -\frac{1}{2\sigma_{\text{eff}}^2} \left(u^2 - 2i\Delta k \sigma_{\text{eff}}^2 u \right) + i\Delta k \bar{\mu}T \\ &= -\frac{1}{2\sigma_{\text{eff}}^2} \left(u - i\Delta k \sigma_{\text{eff}}^2 \right)^2 - \frac{(\Delta k)^2 \sigma_{\text{eff}}^2}{2} + i\Delta k \bar{\mu}T \end{aligned} \quad (106)$$

The term $-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2$ comes out of the integral as a constant factor. This is precisely the **coherence factor** $\exp[-(\Delta k)^2 \sigma_{\text{eff}}^2 / 2]$ that appears in the final formula. Its origin is now clear. It arises from completing the square when we integrate the oscillating cosine against the Gaussian envelope. The faster the oscillations (larger Δk), the more they cancel against the Gaussian, and the smaller the surviving amplitude.

Substituting $v = (u - i\Delta k \sigma_{\text{eff}}^2) / (\sigma_{\text{eff}} \sqrt{2})$, the integral becomes

$$\int \dots du = \sigma_{\text{eff}} \sqrt{2} \exp \left[-\frac{(\Delta k)^2 \sigma_{\text{eff}}^2}{2} + i\Delta k \bar{\mu}T \right] \int_{v_K}^{+\infty} e^{-v^2} dv \quad (107)$$

where the lower limit is $v_K = (x_K - \bar{\mu}T - i\Delta k \sigma_{\text{eff}}^2) / (\sigma_{\text{eff}} \sqrt{2})$. The remaining integral is related to the complementary error function, or equivalently to the normal CDF:

$$\int_{v_K}^{+\infty} e^{-v^2} dv = \frac{\sqrt{\pi}}{2} \operatorname{erfc}(v_K) = \sqrt{\pi} \mathcal{N} \left(\frac{-x_K + \bar{\mu}T + i\Delta k \sigma_{\text{eff}}^2}{\sigma_{\text{eff}}} \right) \quad (108)$$

Collecting everything:

$$J_2 = \sigma_{\text{eff}} \sqrt{2\pi} \exp \left[-\frac{(\Delta k)^2 \sigma_{\text{eff}}^2}{2} + i\Delta k \bar{\mu}T + i(-\Delta\omega T + \varphi_0) \right] \mathcal{N}(d_2^{\text{int}}) \quad (109)$$

with

$$d_2^{\text{int}} = \frac{-x_K + \bar{\mu}T + i\Delta k \sigma_{\text{eff}}^2}{\sigma_{\text{eff}}} \quad (110)$$

This argument is complex, reflecting the oscillatory nature of the model. The normal CDF of a complex argument is well-defined through the analytic continuation of the error function.

Evaluation of I_1

The integral I_1 differs from I_2 by the extra factor e^x in the integrand. This factor shifts the Gaussian. Completing the square with the additional linear term $+x$ is equivalent to shifting the centre from $\bar{\mu}T$ to $\bar{\mu}T + \sigma_{\text{eff}}^2$. Following the same steps as above, the result has the same structure, but with d_2^{int} replaced by

$$d_1^{\text{int}} = \frac{-x_K + \bar{\mu}T + \sigma_{\text{eff}}^2 + i\Delta k \sigma_{\text{eff}}^2}{\sigma_{\text{eff}}} = d_2^{\text{int}} + \sigma_{\text{eff}} \quad (111)$$

This is the interference analogue of the classical relation $d_1 = d_2 + \sigma\sqrt{T}$, with σ_{eff} playing the role of $\sigma\sqrt{T}$.

The extra exponential factor contributes $\exp[\bar{\mu}T + \sigma_{\text{eff}}^2/2 + i\Delta k \sigma_{\text{eff}}^2]$ to the overall coefficient of J_1 compared to J_2 .

Final assembly

Combining I_1 and I_2 , inserting the prefactor \mathcal{P} , and taking the real part, the overlap factor $\mathcal{P} \cdot \sigma_{\text{eff}} \sqrt{2\pi}$ simplifies with the normalisation, and the $\Delta\mu$ penalty combines with the Δk coherence

factor (since $\Delta\mu = \hbar_f \Delta k \sigma^2$, the two Gaussian penalties in Δk can be merged). After bookkeeping, the result is

$$C_{\text{int}} = 2w_1w_2 e^{-rT} e^{-(\Delta k)^2 \sigma_{\text{eff}}^2/2} \left[S_0 e^{\bar{\mu}T} \operatorname{Re}\left(e^{i\Delta\varphi_K} \mathcal{N}(d_1^{\text{int}})\right) - K \operatorname{Re}\left(e^{i\Delta\varphi_K} \mathcal{N}(d_2^{\text{int}})\right) \right] \quad (112)$$

where $\Delta\varphi_K = \Delta k \cdot x_K - \Delta\omega T + \varphi_0$.

This formula has exactly the **Black–Scholes structure**: a term $S_0 \cdot \mathcal{N}(d_1)$ minus a term $K \cdot \mathcal{N}(d_2)$, discounted by e^{-rT} , but modulated by the coherence factor and by the interference phase. When $\Delta k \rightarrow 0$, the phase $\Delta\varphi_K \rightarrow 0$, the real parts reduce to the (real) $\mathcal{N}(d_j)$ and $C_{\text{int}} \rightarrow 2w_1w_2 C_{\text{BS}}$. Added to $w_1^2 C_1 + w_2^2 C_2$, this gives $(w_1 + w_2)^2 C_{\text{BS}} = C_{\text{BS}}$. The classical price is recovered, as it must be.

A note on the complex normal CDF

For numerical implementation, the normal CDF of a complex argument $z = a + ib$ is computed via the Faddeeva function $w(z) = e^{-z^2} \operatorname{erfc}(-iz)$:

$$\mathcal{N}(z) = \frac{1}{2} \operatorname{erfc}\left(-\frac{z}{\sqrt{2}}\right) \quad (113)$$

Efficient algorithms for the Faddeeva function are available in standard numerical libraries (SciPy's `scipy.special.wofz`, MATLAB's `erfcx` with complex arguments). For moderate values of $|\Delta k| \sigma_{\text{eff}}$ — the typical regime of interest — the imaginary part of the argument is not too large, and $\operatorname{Re}(\mathcal{N}(z))$ oscillates gently around the real-valued $\mathcal{N}(a)$, confirming the interpretation of C_{int} as an oscillatory correction to the Black–Scholes price.

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